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Exercises

Engineering Signal Analysis

from Fourier to filtering

TU Delft OPEN Publishing

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– from Fourier to filtering –

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Contents

Preface

This book, **Exercises**, serves as the companion of the textbook **Theory**, which provides an introduction to the subject of signal analysis in the frequency domain. The Exercises book follows the same structure: five parts containing material for the corresponding 18 chapters. There are no separate exercises for the material presented in the appendices of the textbook.

In an engineering curriculum we attach great value to application of the theory. To become proficient in signal analysis, a relatively abstract, theoretical subject, it is important to practice and discover the world behind the mathematics. This book contains many exercises, ranging from completing small theoretical proofs, performing derivations, working out numerical calculation problems to doing computer-based assignments with Python. All exercises are accompanied by (comprehensive) solutions, to facilitate learning the material and extend insight and understanding.

With Python, the reader can carry out advanced and more elaborate exercises. It allows for quick and easy visualization of signals in the time and frequency domains, which substantially contributes to the learning process. The scripts included in this book (available online via data.4tu.nl) contain only basic Python code aimed at obtaining the requested numerical and graphical results. More advanced code for further optimization and layout (as, for instance, LaTeX symbol typesetting) has intentionally been omitted.

The authors have taught this material for many years using the book 'Signals & Systems – Continuous and Discrete' [1]. The engineering approach in that book, combining theory with many practical examples and exercises, has been an inspiration for the present book.

The authors welcome corrections and suggestions for improvement, as well as feedback in general. Please e-mail ESAbook-TUD@tudelft.nl for this purpose.

The authors would like to thank Sebastiaan de Stigter (cyanetica.com) for his beautiful book cover design, and the TU Delft OPEN Publishing team for their support with the publication process.

Max Mulder, Rowenna Wijlens, Christian Tiberius
Delft, April 2026

I

Introduction

1

Introduction

Exercises

e | 1.2 Signals in time and frequency

e.1-1

In this first exercise we return to the model $\tilde{x}(t)$ of a periodic signal $x(t)$, (1.1) on page 5 of the companion book, Theory. The period of signal $x(t)$ is set to $T_0 = 2$ s, the same as in Figure 1.3. The amplitudes of the signal model also match those shown in the magnitude plot A_k as a function of frequency, and corresponding index k, in Figure 1.3; they are tabulated below for $k = 0, 1, 2, \ldots, 9$. What differs, however, are the phases θ_k in the model $\tilde{x}(t)$, see below. We will investigate what happens when constructing the signal model $\tilde{x}(t)$ with these adjusted phases.

k	0	1	2	3	4	5	6	7	8	9
A_k	0	$\frac{4}{\pi}$	0	$\frac{4}{3\pi}$	0	$\frac{4}{5\pi}$	0	$\frac{4}{7\pi}$	0	$\frac{4}{9\pi}$
θ_k		0	0	π	0	0	0	π	0	0

(a) Sketch the first two non-zero building blocks of $\tilde{x}(t)$ for $-2 \leq t \leq 2$ s:

$$b_1(t) = A_1 \cos(2\pi 1 f_0 t + \theta_1) \quad \text{and} \quad b_3(t) = A_3 \cos(2\pi 3 f_0 t + \theta_3)$$

(b) Sketch the sum of these two building blocks. What do you think would happen to the signal model $\tilde{x}(t)$ when adding more non-zero building blocks, e.g., for $k = 5, 7, 9$? Do we ultimately get a square wave, similar to Figure 1.3, and if so, how do the two square waves differ from each other, if at all?

(c) What would happen to the signal model $\tilde{x}(t)$ obtained in (b) when we would set $A_0 = 1$?

e| 1.c Computer exercises

e.1-2

When performing Exercise *e*.1-1 it quickly becomes clear that it is easier to write a Python script and have the computer do the work. That is exactly what we recommend you to do here: create a script that implements (1.1), then perform the following tasks, for $-2 \leq t \leq 2$ s.

(a) Set $A_0 = 0$ and $T_0 = 2$ s. Create five non-zero building blocks $b_k(t)$, for $k = 1, 3, 5, 7, 9$, using the amplitudes and phases from Exercise *e*.1-1, and add them together. Show the five building blocks and their sum as six subplots in one figure.

(b) Repeat (a), but this time with $A_0 = -1$ and $T_0 = 1$ s. Explain the differences you observe with respect to (a).

(c) Run the script with the amplitudes and phases as tabulated below, with $T_0 = 1$ s. Describe the periodic waveform that you obtain.

k	0	1	2	3	4	5	6	7	8	9
A_k	$\frac{2}{\pi}$	0	$-\frac{4}{3\pi}$	0	$-\frac{4}{15\pi}$	0	$-\frac{4}{35\pi}$	0	$-\frac{4}{63\pi}$	0
θ_k		0	0	0	0	0	0	0	0	0

Solutions

s | 1.2 Signals in time and frequency

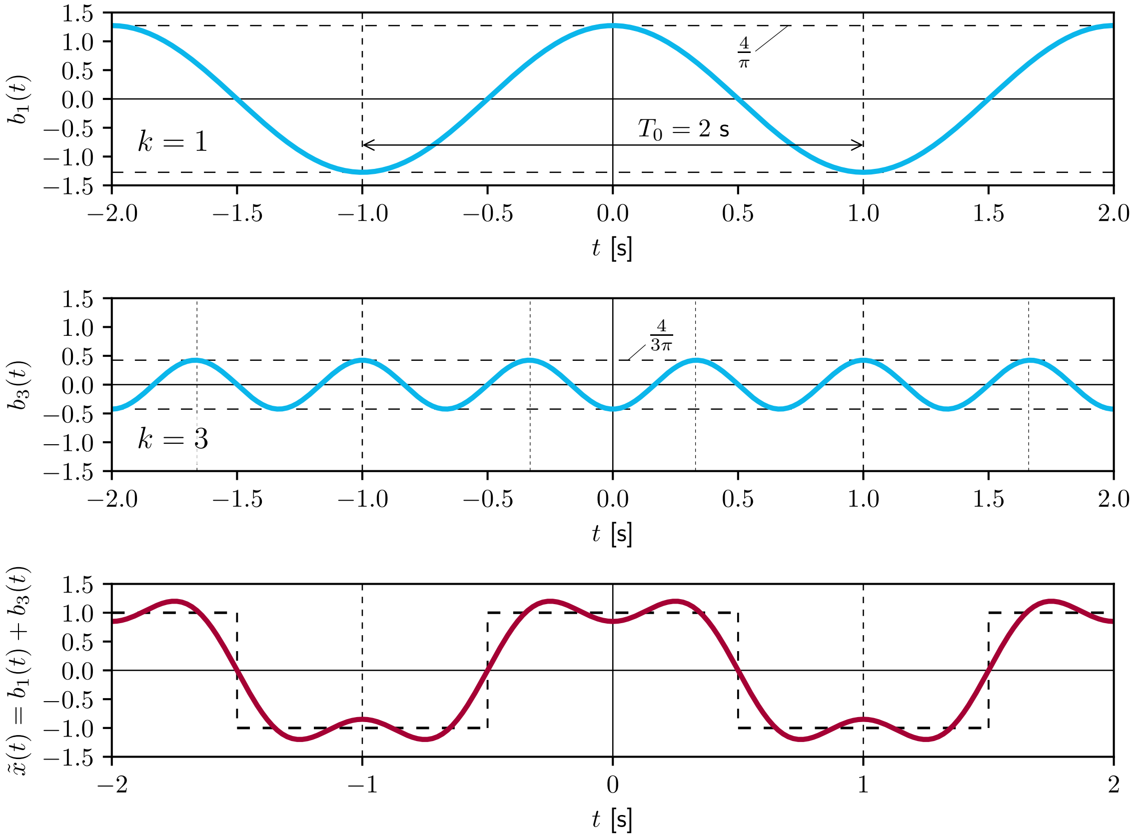
s.1-1

(a) The figure below shows building blocks $b_1(t)$ and $b_3(t)$, in blue, for $-2 \leq t \leq 2$ s. The first component, for $k = 1$ (first harmonic) and with $f_0 = \frac{1}{2}$ Hz, is:

$$b_1(t) = A_1 \cos(2\pi f_0 t + \theta_1) = \frac{4}{\pi} \cos\left(2\pi \frac{1}{2} t + 0\right) = \frac{4}{\pi} \cos(\pi t)$$

For component $b_3(t)$, with $k = 3$ (third harmonic) and $kf_0 = \frac{3}{2}$ Hz, we obtain:

$$b_3(t) = A_3 \cos(2\pi 3 f_0 t + \theta_3) = \frac{4}{3\pi} \cos\left(2\pi 3 \frac{1}{2} t + \pi\right) = -\frac{4}{3\pi} \cos(3\pi t)$$



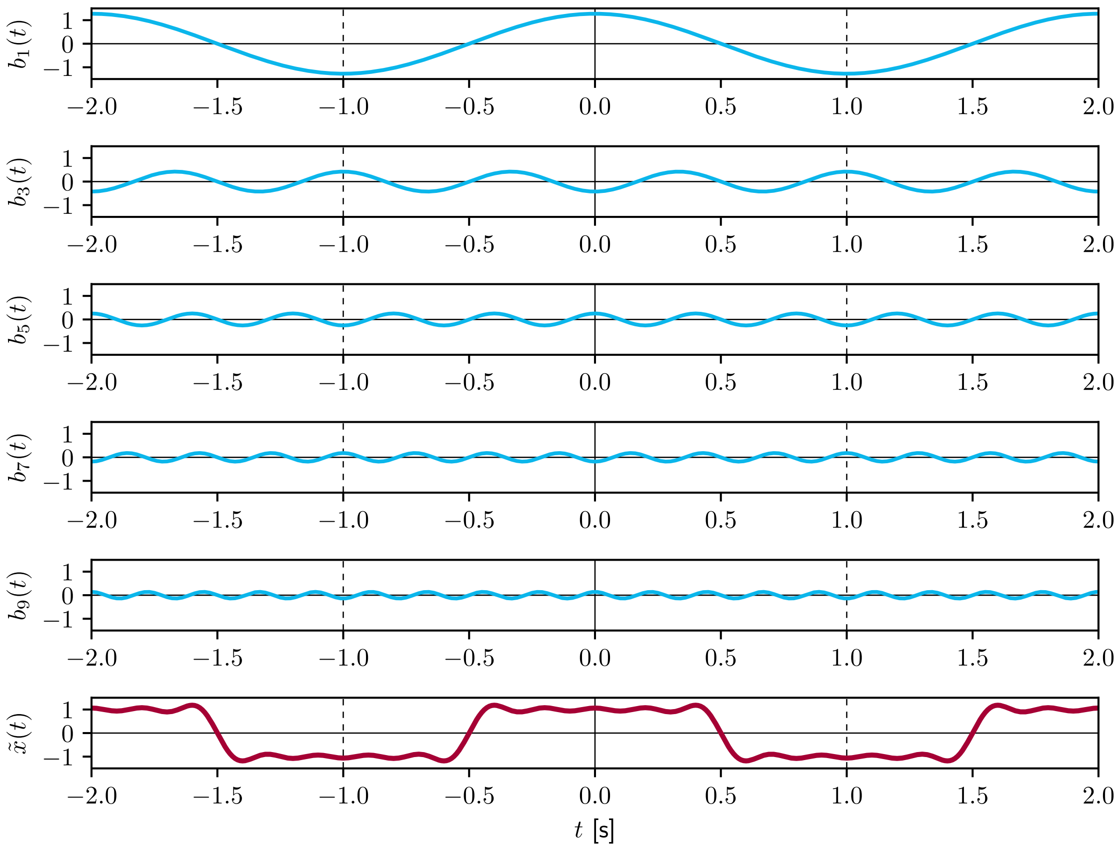
(b) The sum of $b_1(t)$ and $b_3(t)$ is shown above, in burgundy. Adding more non-zero building blocks ($b_5(t) = A_5 \cos(2\pi 5 f_0 t + \theta_5)$ et cetera) will lead to a model $\tilde{x}(t)$ that better approximates a square wave (shown by the black dashed lines in the bottom graph), similar to Figure 1.3. Because some of the phases of the model are different, however, than those used to construct the model shown in Figure 1.3, the model here builds a square wave that is shifted by a quarter period to the left and is symmetric about $t = 0$: we obtain an even function of time, i.e., $\tilde{x}(-t) = \tilde{x}(t)$, explained in detail in Section 2.2.

(c) When $A_0 = 1$, all building blocks $b_k(t)$ remain the same. But because the model now has a non-zero A_0, (1.1) indicates that $\tilde{x}(t)$ will be shifted upward by 1, and the square wave model will oscillate between 0 and 2, about $A_0 = 1$. The period of the wave does not change.

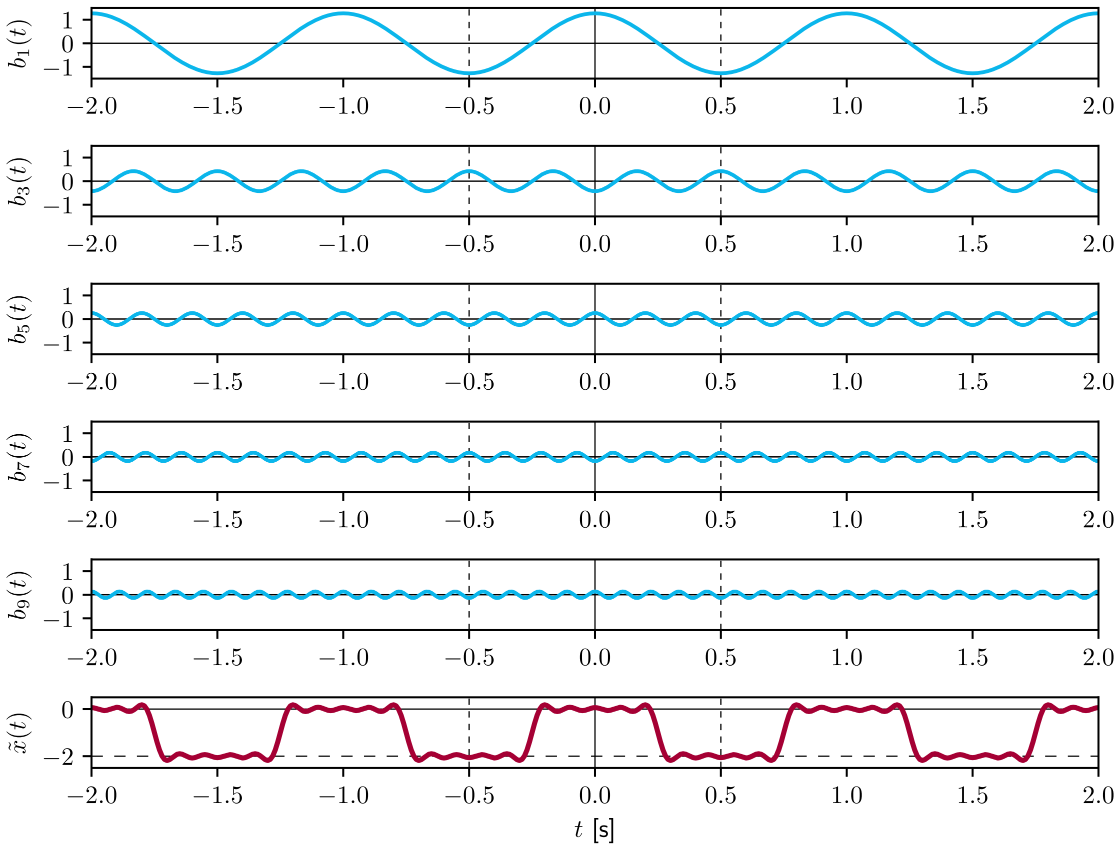
s| 1.c Computer exercises

S.1-2

(a)

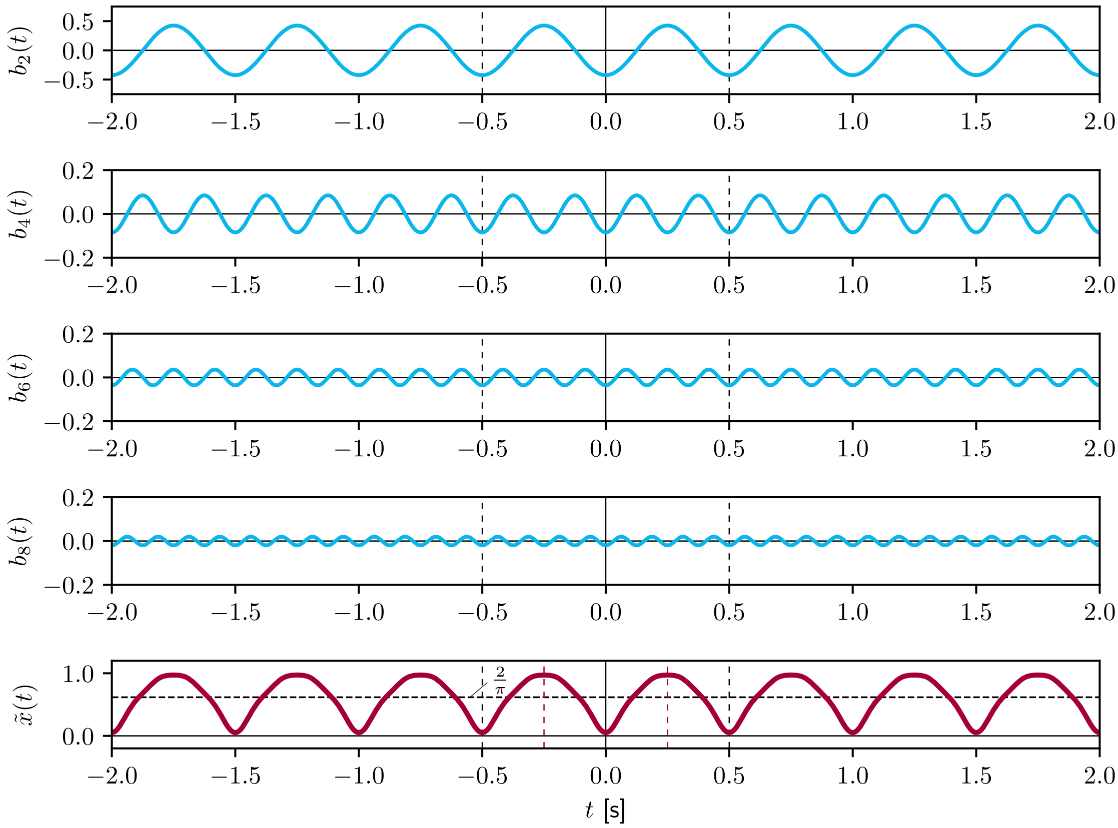


(b)



With respect to (a) the signal model period is halved, the fundamental frequency f_0 doubles, the signal model oscillates twice as fast in time. Because $A_0 = -1$, $\tilde{x}(t)$ is shifted down by 1 and oscillates between 0 and -2. With signal models constructed using (1.1), the constructed signal will always oscillate about the value for A_0.

(c)



The periodic waveform obtained is a model of the rectified sine wave, $x(t) = |\sin(2\pi t)|$, an even signal in time, see Section 2.2.

Note that $A_0 = \frac{2}{\pi}$, which causes the sum of the non-zero building blocks, i.e., for $k = 2, 4, 6, 8$, to oscillate about $\frac{2}{\pi}$ (dashed horizontal line in the above figure at bottom). This latter sum, by itself, always oscillates about 0. A non-zero A_0 coefficient in (1.1) shifts the modeled signal $\tilde{x}(t)$ up or down. The oscillation about A_0 is, in this case, not symmetric.

Note that when setting $T_0 = 1$ s in our script we have $f_0 = 1$ Hz in the $\tilde{x}(t)$-model building blocks, (1.1). All harmonics in this model are an integer multiple of f_0. The signal that the model describes, however, has a fundamental frequency of 2 Hz! This is because while the sine wave in the description of $x(t)$ has a frequency of 1 Hz, and repeats itself every second, rectifying it leads to a signal which repeats itself every $\frac{1}{2}$ s. For this reason, all building blocks with odd indices ($k = 1, 3, 5, \ldots$) in the signal model are zero.

Basic Python code for (a):

```
import numpy as np
from matplotlib import pyplot as plt

burgundy = [165/255,0/255,52/255]        # burgundy color
```

```
# time vector: time steps 0.01 s from -2 s to 2 s
Tend = 2; t = np.linspace(-Tend, Tend, 401)

# five odd-indexed building blocks b_k(t) with A0 = 0 and T0 = 2 s
T0 = 2                  # period T0 [s]
A0 = 0                  # constant A0

# note that the even-indexed building blocks b_k(t) are zero
# signal components, amplitudes
A_1 = 4/np.pi; A_3 = A_1/3; A_5 = A_1/5; A_7 = A_1/7; A_9 = A_1/9
# signal components, phases
theta_1 = 0; theta_3 = np.pi; theta_5 = 0; theta_7 = np.pi; theta_9 = 0
# signal components, frequencies
f0 = 1/T0; f_1 = 1*f0; f_3 = 3*f0; f_5 = 5*f0; f_7 = 7*f0; f_9 = 9*f0
# odd-indexed building blocks b_k(t)
b_1 = A_1 * np.cos(2 * np.pi * f_1 * t + theta_1)
b_3 = A_3 * np.cos(2 * np.pi * f_3 * t + theta_3)
b_5 = A_5 * np.cos(2 * np.pi * f_5 * t + theta_5)
b_7 = A_7 * np.cos(2 * np.pi * f_7 * t + theta_7)
b_9 = A_9 * np.cos(2 * np.pi * f_9 * t + theta_9)

# signal model x~(t) consisting of summed A0 and building blocks b_k(t)
x = A0 + b_1 + b_3 + b_5 + b_7 + b_9

# plot five odd-indexed building blocks b_k(t) and their sum
plt.figure()
plt.subplot(6,1,1)
plt.plot(t, b_1, color='b', linewidth=1.5); plt.ylabel('$b_1(t)$')
plt.subplot(6,1,2)
plt.plot(t, b_3, color='b', linewidth=1.5); plt.ylabel('$b_3(t)$')
plt.subplot(6,1,3)
plt.plot(t, b_5, color='b', linewidth=1.5); plt.ylabel('$b_5(t)$')
plt.subplot(6,1,4)
plt.plot(t, b_7, color='b', linewidth=1.5); plt.ylabel('$b_7(t)$')
plt.subplot(6,1,5)
plt.plot(t, b_9, color='b', linewidth=1.5); plt.ylabel('$b_9(t)$')
plt.subplot(6,1,6)
plt.plot(t, x, color=burgundy, linewidth=2)
plt.xlabel('$t$ [s]'); plt.ylabel(r'$\tilde{x}(t)$')

# the remainder of the code for (b) and (c) is highly similar to the code
# above for (a) and is therefore omitted to avoid excessive repetition
```

2

Signal preliminaries

Exercises

e|2.1 Signal models

Note: the principle of quantization is explained in Appendix D of the companion book, Theory. Hence, for more guidance on how to solve Exercises *e*.2-1 and *e*.2-2, one is encouraged to read Appendix D.

***e*.2-1**

We would like to quantize an analog voltage signal

$$x(t) = 2 - 1.5\sin^2\left(15\pi t + \frac{\pi}{6}\right)$$

with a step size Δx of at most 0.05 V.

Using a binary representation, how many bits do we need for each sample, such that the step size requirement is just met?

***e*.2-2**

A light sensor delivers a voltage to the analog-to-digital converter in the range $[0.0, 5.0]$ V. The converter uses a binary representation of the signal level and uses 10 bits to quantize the incoming signal.

What is the *maximum* quantization error?

***e*.2-3**

Determine the phase shift θ (in [rad]) and time shift t_0 (in [s]) of all signals given below. Remember, the zero-phase, positive-amplitude cosine is our reference for defining the phase and time shift of a signal, see page 15 of the companion book, Theory.

(a) $x_a(t) = 4\cos(2\pi t + \pi)$

(b) $x_b(t) = \sin(5\pi t)$

(c) $x_c(t) = \pi\cos(36\pi t + 12)$

(d) $x_d(t) = \sin(2\pi t + 2)$

(e) $x_e(t) = -\frac{3}{2}\cos(\pi t)$

(f) $x_f(t) = 2\cos^2\left(2\pi t + \frac{\pi}{6}\right)$

e.2-4 Consider a signal $x(t)$, which consists of two sinusoidal components with frequencies $f_1 = 2.4$ Hz and $f_2 = 8.4$ Hz, respectively.
What are the fundamental frequency f_0 and period T_0 of $x(t)$?

e.2-5 What are the fundamental frequencies f_0 and periods T_0 of the signals given below?

(a) $x_a(t) = \sin(20\pi t)$

(b) $x_b(t) = \cos(30\pi t)$

(c) $x_c(t) = \cos(36\pi t)$

(d) $x_d(t) = x_a(t) + x_b(t)$

(e) $x_e(t) = x_a(t) + x_c(t)$

(f) $x_f(t) = x_b(t) + x_c(t)$

e.2-6 What are the fundamental frequencies f_0 and periods T_0 of the signals given below?

(a) $x_a(t) = 4\cos\left(5\pi t - \frac{\pi}{3}\right)$

(b) $x_b(t) = 3\cos\left(9\pi t + \frac{\pi}{6}\right)$

(c) $x_c(t) = 5\sin\left(11\pi t - \frac{\pi}{4}\right)$

(d) $x_d(t) = x_a(t) + x_b(t)$

(e) $x_e(t) = x_a(t) + x_c(t)$

(f) $x_f(t) = x_b(t) + x_c(t)$

e.2-7 Determine for each of the signals given below whether it is periodic.
If so, what are its fundamental frequency f_0 and period T_0?

(a) $x_a(t) = 2\sin\left(\frac{3\pi}{4}t + \frac{\pi}{4}\right) + 7\sin\left(4\pi t - \frac{\pi}{6}\right)$

(b) $x_b(t) = -\frac{1}{2} + \sin^2(2\pi t) - \frac{1}{2}\sin\left(4\pi t - \frac{\pi}{2}\right)$

(c) $x_c(t) = -3 + \cos(15\pi t) + \sin\left(15\pi t - \frac{\pi}{2}\right) + 4\cos\left(30\pi t + \frac{\pi}{16}\right) - 5\sin\left(20\pi t + \frac{\pi}{3}\right)$

Note: one is encouraged to read Appendix Section B.4.3 of the companion book, Theory, on the Dirac delta function before getting started with Exercises e.2-8 and e.2-9.

e.2-8 Evaluate the following integrals:

(a) $\displaystyle\int_{-\infty}^{\infty} e^{-\alpha t^2}\,\delta(t-5)\,\mathrm{d}t$

(b) $\displaystyle\int_{0}^{\infty} e^{-\alpha t^2}\,\delta(t+5)\,\mathrm{d}t$

(c) $\displaystyle\int_{-\infty}^{\infty} \big(3 + \cos(3\pi t)\big)\,\delta(t)\,\mathrm{d}t$

e.2-9 Evaluate the following integrals:

(a) $\displaystyle\int_{0}^{6} \cos(4\pi t)\,\delta(t-4)\,\mathrm{d}t$

(b) $\displaystyle\int_{6}^{12} \cos(4\pi t)\,\delta(t-4)\,\mathrm{d}t$

(c) $\displaystyle\int_{0}^{6} \cos(4\pi t)\,\delta(t-0.75)\,\mathrm{d}t$

(d) $\displaystyle\int_{-\infty}^{\infty} (t-4)^2\,\delta(t-4)\,\mathrm{d}t$

(e) $\displaystyle\int_{-\infty}^{\infty} t^2\,\delta(t-4)\,\mathrm{d}t$

(f) $\displaystyle\int_{-\infty}^{\infty} t^4\,\delta(t-4)\,\mathrm{d}t$

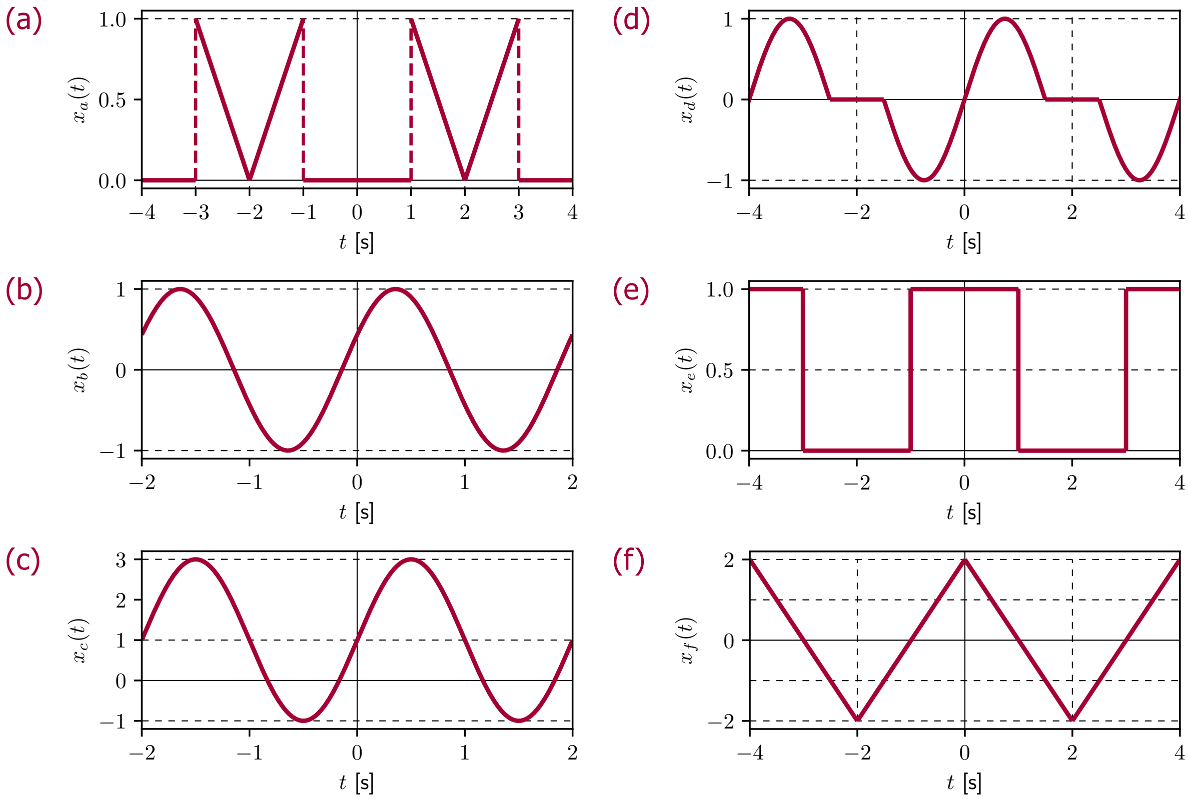
e|**2.2** Signal symmetry

e.2-10

In Section 2.2 the three signal symmetry properties are defined: even, odd and half-wave odd. In some books and references a fourth symmetry property is mentioned: *quarter-wave* symmetry. A signal $x(t)$ is said to have quarter-wave symmetry when this signal has half-wave odd symmetry *and* is either even or odd. For the six signals illustrated below, of which two periods are shown:

(1) Compute the average of the signal.

(2) Comment on the signal symmetry, following the mathematical definitions of Table 2.1, and including the quarter-wave symmetry property.



e.2-11

In Section 2.2 the three signal symmetry properties are defined: even, odd and half-wave odd. We may ask ourselves: is it also possible for a signal to have half-wave *even* symmetry? Before we start, recall that a signal is defined to have half-wave odd symmetry when three conditions are met:

1. The signal is periodic (period T_0).
2. The average of the signal is zero.
3. The following property holds: $x(t \pm \frac{T_0}{2}) = -x(t)$.

(a) What would be the conditions necessary for a signal to be half-wave even, similar to the conditions above for half-wave odd signals? In particular, try to formulate condition number 3 above for a half-wave even signal.

(b) Comment on the fallacy of the half-wave even condition derived in (a).

e|2.3 Sinusoid building block

Note: one is encouraged to read Appendix C of the companion book, Theory, on complex algebra before getting started with the exercises in this section.

e.2-12

Consider the two complex numbers $A = 4 + j4$ and $B = 12e^{j\frac{\pi}{6}}$.

(a) Put A into polar form and B into Cartesian form. What are ...

(1) ... the magnitude and angle (or argument) of A?

(2) ... the real and imaginary parts of B?

(b) Compute the sum of A and B. Show the sum as a vector in the complex plane along with A and B.

(c) Compute the difference of A and B. Show the difference as a vector in the complex plane.

(d) Compute the product of A and B twice, once with both numbers expressed in

(1) Cartesian form, and once in (2) polar form.

Show that both answers are equivalent.

(e) Compute the quotient A/B twice, once with both numbers expressed in

(1) Cartesian form, and once in (2) polar form.

Show that both answers are equivalent.

e.2-13

Derive Euler's formula (C.1) using the following Taylor series:

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \ldots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

e.2-14

Consider the three periodic signals defined below. For each of these signals:

(1) Determine its fundamental frequency f_0 and which harmonics are present in the signal.

(2) Describe the signal using the real part of the complex-valued signal description using phasors, as in (2.9).

(3) Draw its phasors as vectors in the complex plane.

For the phasors, use the notation $\vec{X}_k$ as introduced in (C.11), with subscript k referring to the kth harmonic present in the signal.

(a) $x_a(t) = 4\cos\left(2\pi t + \frac{\pi}{4}\right) - 2\sin\left(14\pi t - \frac{\pi}{2}\right)$

(b) $x_b(t) = 1 - 2\sin\left(2\pi t + \frac{\pi}{4}\right) + 4\cos\left(5\pi t - \frac{\pi}{2}\right)$

(c) $x_c(t) = -2 + 4\sin(4\pi t) + 3\cos\left(4\pi t + \frac{\pi}{4}\right)$

e.2-15 Consider the three periodic signals from Exercise e.2-14. For each of these signals:

(1) Obtain its complex conjugate phasor pair signal description, as in (2.10).

(2) Draw its complex conjugate phasor pairs as vectors in the complex plane.

For the complex conjugate phasor pairs, use the notation $\{\frac{1}{2}\vec{X}_k, \frac{1}{2}\vec{X}_{-k}\}$, with subscript k referring to the kth harmonic present in the signal.

Regarding (1), in the signal description, write out the complex numbers corresponding to the phasors in full.
Regarding (2), recall that the complex exponential basis function with the positive frequency kf_0 is multiplied by the phasor $\vec{X}_k$, divided by two, i.e., $\frac{1}{2}\vec{X}_k$; the latter can be indicated with a '∘'-symbol in your sketch. The complex exponential basis function with the negative frequency $-kf_0$ is multiplied by the complex conjugate of phasor $\vec{X}_k$ ($\vec{X}_k^* = \vec{X}_{-k}$), divided by two, i.e., $\frac{1}{2}\vec{X}_{-k}$; the latter can be indicated with a '⋆'-symbol.

e.2-16 Consider the signals from Exercise e.2-6.

(a) Describe them using the real part of the complex-valued signal description using phasors.

(b) Obtain their complex conjugate phasor pair signal description.

(c) Sketch their single-sided amplitude and phase spectra.

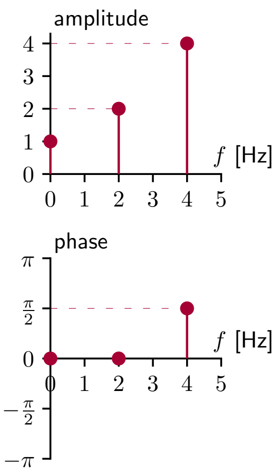
(d) Sketch their double-sided amplitude and phase spectra.

e.2-17 Consider the signal

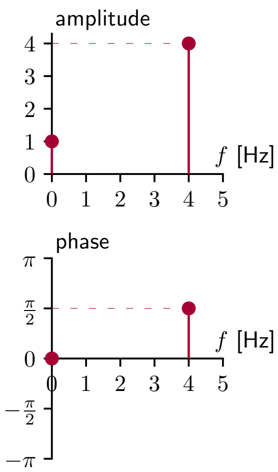
$$x(t) = 2\cos^2(2\pi t) + 4\cos\left(8\pi t + \frac{\pi}{2}\right) + \sin\left(4\pi t - \frac{\pi}{2}\right).$$

Below four possible versions of the signal's *single*-sided amplitude and phase spectra are shown, of which only one is correct. Each numbered version shows the amplitude spectrum at the top and the corresponding phase spectrum at the bottom. Which one is correct?

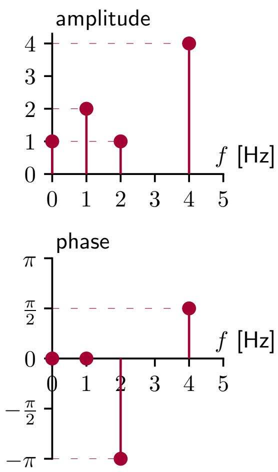
(I)



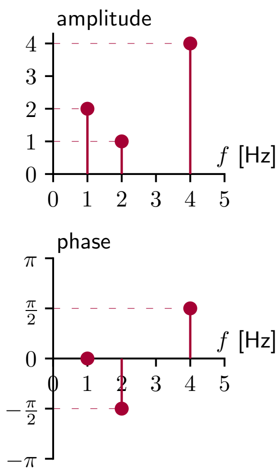
(II)



(III)



(IV)



e.2-18 Consider the signal

$$x(t) = 3\cos\left(4\pi t - \frac{\pi}{6}\right) + 5\sin(14\pi t).$$

(a) Is it periodic? If so, what are its fundamental frequency f_0 and period T_0?

(b) Describe it using the real part of the complex-valued signal description using phasors.

(c) Sketch its single-sided amplitude and phase spectra.

(d) Obtain its complex conjugate phasor pair signal description.

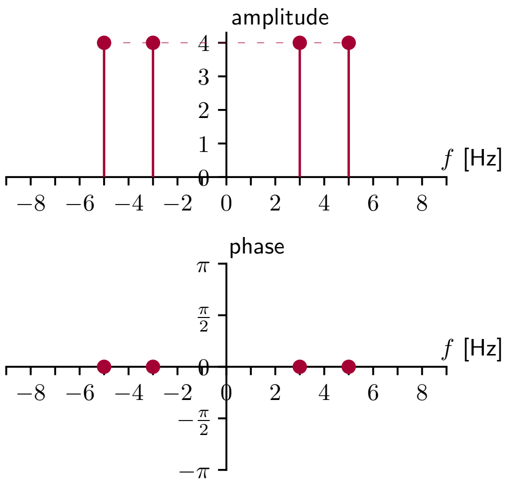
(e) Sketch its double-sided amplitude and phase spectra.

e.2-19 Consider the signal

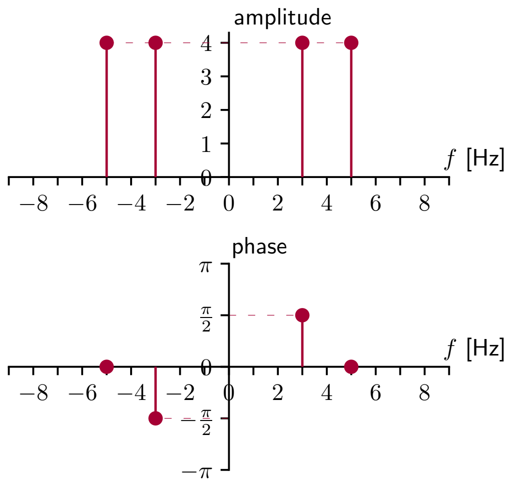
$$x(t) = 8\sin\left(6\pi t + \frac{\pi}{2}\right)\cos(10\pi t).$$

Below four possible versions of the signal's *double*-sided amplitude and phase spectra are shown, of which only one is correct. Each numbered version shows the amplitude spectrum at the top and the corresponding phase spectrum at the bottom. Which one is correct?

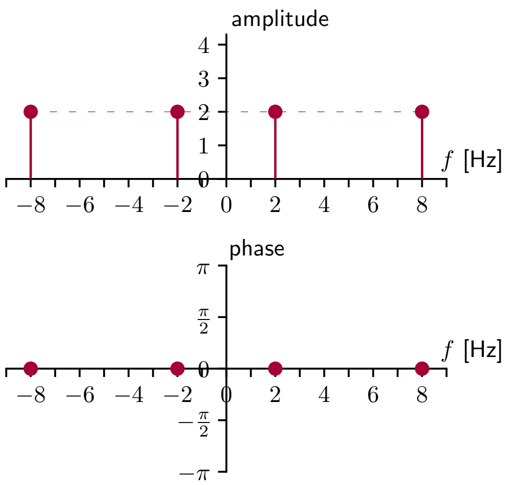
(I)



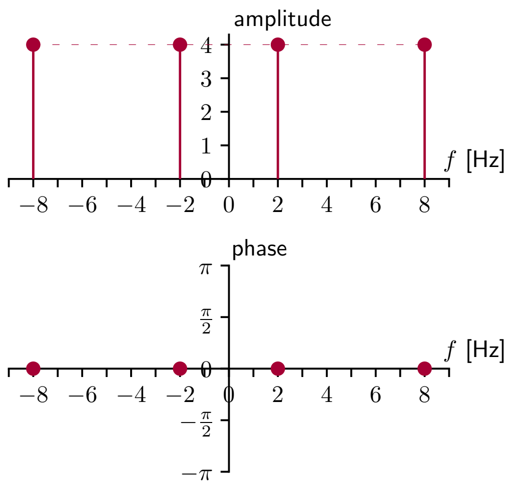
(II)



(III)



(IV)



e.2-20

Consider the signal

$$x(t) = \sin^2\left(5\pi t - \frac{\pi}{4}\right) + 2\cos\left(2\pi t - \frac{\pi}{2}\right).$$

(a) Describe it using the real part of the complex-valued signal description using phasors.

(b) Sketch its single-sided amplitude and phase spectra.

(c) Obtain its complex conjugate phasor pair signal description.

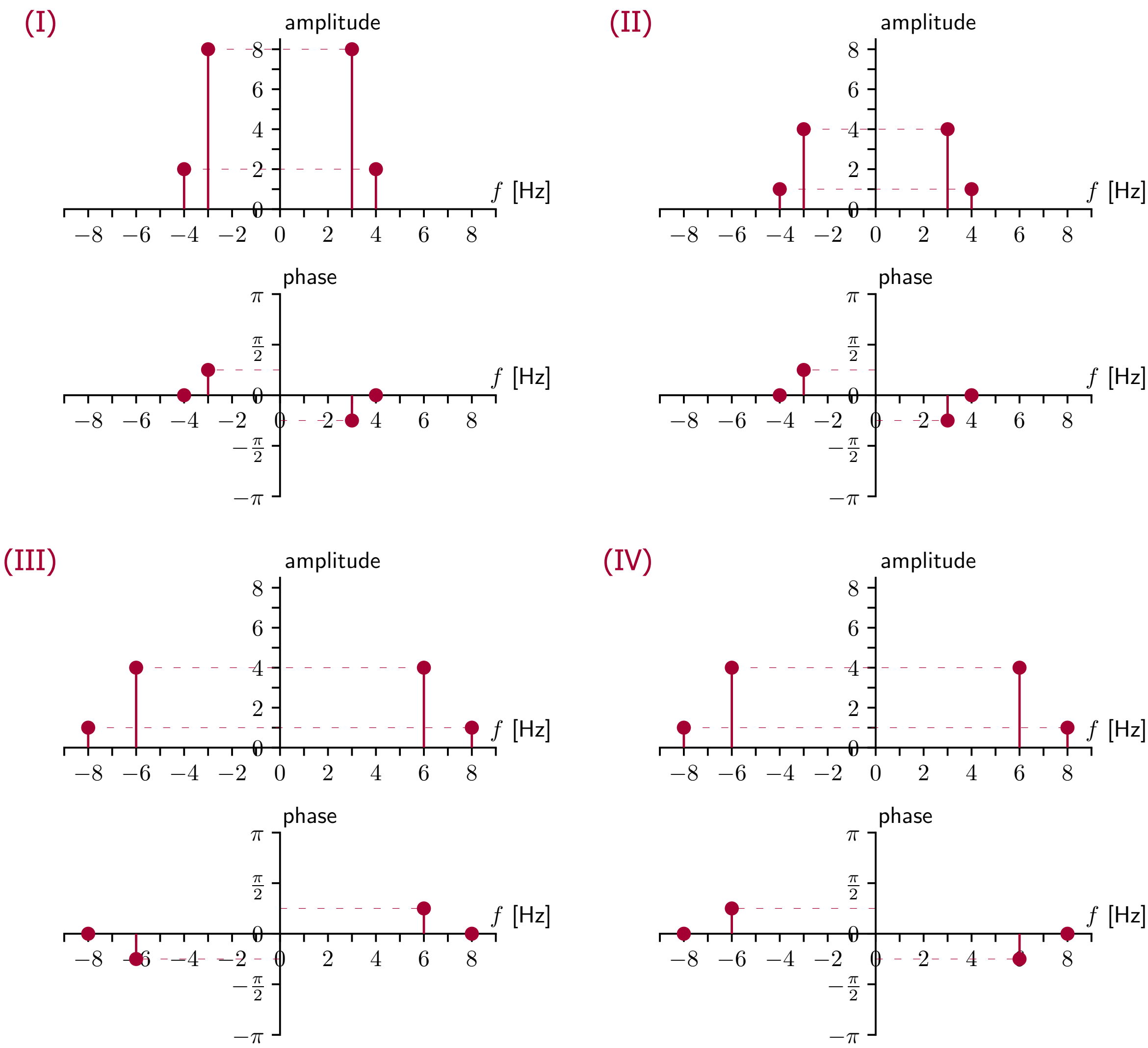
(d) Sketch its double-sided amplitude and phase spectra.

e.2-21

Consider the signal

$$x(t) = -2 + 4\cos^2(4\pi t) + 8\sin\left(6\pi t + \frac{\pi}{4}\right).$$

Below four possible versions of the signal's *double*-sided amplitude and phase spectra are shown, of which only one is correct. Each numbered version shows the amplitude spectrum at the top and the corresponding phase spectrum at the bottom. Which one is correct?



e|2.4 Signal operations

e.2-22 Sketch the following signals:

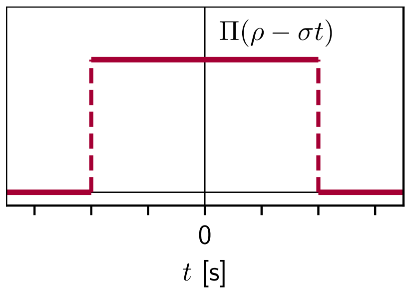
(a) $\Pi(0.2t)$ (c) $\Pi\left(t-\frac{1}{5}\right)$ (e) $\Pi\left(\frac{t-1}{6}\right)+\Pi(t-1)$

(b) $\Pi(20t)$ (d) $\Pi\left(\frac{t-3}{4}\right)$ (f) $\Pi\left(\frac{t-2}{4}\right)-\Pi\left(\frac{t-2}{3}\right)$

e.2-23 Consider unit pulse signal $\Pi(\rho-\sigma t)$, with parameters ρ and σ, which is illustrated here on the right.

For both ρ and σ *individually*, choose the correct statement:

(1) The parameter is smaller than zero (parameter < 0).
(2) The parameter is larger than zero (parameter > 0).
(3) The parameter is equal to zero (parameter $= 0$).
(4) The parameter's sign cannot be determined.



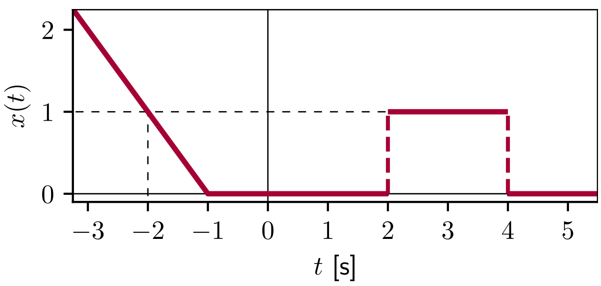
e.2-24 Sketch the following signals:

(a) $\Pi(3t+4)$ (c) $u\left(\frac{t-1}{2}\right)$ (e) $r\left(\frac{t+3}{2}\right)$

(b) $\Pi\left(-\frac{1}{2}t+1\right)$ (d) $u(-2t+3)$ (f) $r(-4t+1)$

e.2-25 Consider signal $x(t)$ illustrated on the right. This signal is a combination of an (operated) unit ramp function $r(t)$ and an (operated) unit pulse function $\Pi(t)$.

Provide the expression for $x(t)$.



e.2-26 In this exercise, four different periodic signals $x(t)$ are given, with different periods T_0. For each signal, we create signal $y(t)$ that equals signal $x(t)$ but is shifted a quarter period (or wavelength) to the left: $y(t)=x(t+\frac{T_0}{4})$. For each of these signals:

(1) Determine the fundamental period T_0 and fundamental frequency f_0.
(2) Solve for the unknown variable(s).
(3) Determine the phase shift(s) θ and time shift t_0 of $y(t)$ with respect to $x(t)$.

(a) $x_a(t)=2\sin(2\pi t)$, $y_a(t)=2\sin(2\pi t+a)$

(b) $x_b(t)=-3\cos\left(3\pi t+\frac{\pi}{3}\right)$, $y_b(t)=-3\cos\left(3\pi t+\frac{\pi}{3}+b\right)$

(c) $x_c(t)=4\sin\left(5\pi(t+2)\right)$, $y_c(t)=4\sin\left(5\pi(t+2)+c\right)$

(d) $x_d(t)=2\cos\left(6\pi t+\frac{\pi}{4}\right)-3\sin(15\pi t)$, $y_d(t)=2\cos\left(6\pi t+\frac{\pi}{4}+d_1\right)-3\sin(15\pi t+d_2)$

e.2-27

Consider the signal

$$x(t) = \Pi\left(\frac{t-4}{5}\right) + \Pi\left(\frac{t-5}{2}\right).$$

(a) Sketch $x(t)$.

(b) Express $x(t)$ in terms of step functions.

(c) Express the derivative of $x(t)$ in terms of unit impulses.

e|**2.5** Energy and power

e.2-28

Sketch the following signals and calculate their energies:

(a) $e^{-5t}u(t)$

(b) $u(t) - u(t-12)$

(c) $\cos(5\pi t)\,u(t)u(4-t)$

(d) $2r(t-3) + 2r(t+1) - 4r(t-1)$

e.2-29

Obtain the average powers of the signals given in Exercise e.2-5 on page 10.

e.2-30

Obtain the average powers of the signals given in Exercise e.2-6 on page 10.

e.2-31

Determine for each of the following signals whether it is an energy signal, a power signal, or neither. If it is an energy or power signal, calculate its energy or average power.

(a) $u(t+1) + 4u(t-2) - 3u(t-4)$

(b) $u(t+1) + 4u(t-2) - 5u(t-4)$

(c) $4e^{-8t}u(t)$

(d) $\left(4e^{-8t} + 3\right)u(t)$

(e) $\left(3 - 4e^{-8t}\right)u(t)$

(f) $0.1r(t)$

(g) $0.1r(t) - 0.1r(t-5)$

(h) $t^{-\frac{1}{6}}u(t-6)$

e.2-32

Consider the signal from Exercise e.2-18 on page 14:

$$x(t) = 3\cos\left(4\pi t - \frac{\pi}{6}\right) + 5\sin(14\pi t)$$

Determine whether it is an energy signal, a power signal, or neither. If it is an energy or power signal, calculate its energy or average power.

e.2-33

Determine for each of the following signals whether it is an energy signal or a power signal. Compute its energy or average power.

(a) $2u(t+2) - 2u(t-3)$

(b) $2u(t) - \frac{2}{3}u(t-5)$

(c) $2te^{-3t}u(t)$

(d) $r(t) - r(t-2) - r(t-6) + r(t-8)$

(e) $r(t+1) - r(t-1)$

(f) $r(t) - 2r(t-1) + 2r(t-3) - r(t-4)$

(g) $4\left(r(t+1) - 2r(t) + r(t-2) + 1\right)$

(h) $8\cos\left(6\pi t + \frac{\pi}{3}\right) + r(t+3) - 2r(t) + r(t-3)$

e.2-34 Answer the following questions for each of the signals given below:

(1) Is the signal periodic? If so, give its period.

(2) Is it an energy signal? If so, compute its energy.

(3) Is it a power signal? If so, compute its average power.

(a) $\cos(3\pi t) + \sin(4\pi t)$

(b) $\sin(5t) + \cos(5\pi t)$

(c) $3e^{-2t}u(t)$

(d) $e^{5t}u(t)$

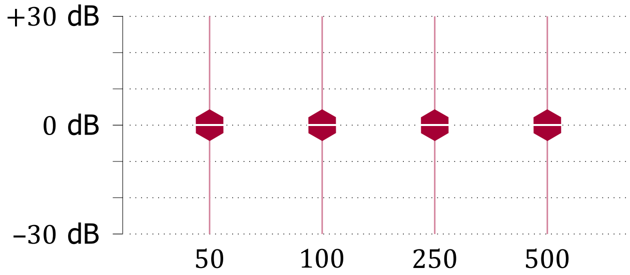
e.2-35 (a) Express a signal power of 1 [microWatt] in decibel.

(b) When the power of a signal, expressed in Watt, is doubled, how many dB in terms of dB W are added?

e.2-36 The voltage signal

$$x(t) = 2\cos\left(100\pi t + \frac{\pi}{3}\right) + 4\sin(200\pi t) - 2\cos\left(400\pi t - \frac{\pi}{6}\right) - \cos(1000\pi t)$$

is fed to a (graphical) equalizer, illustrated below.



The equalizer works as follows: it has an input signal, $x(t)$, and an output signal, $y(t)$. At its four frequencies (indicated at the bottom of the illustration), the equalizer passes the input signal to the output. Other frequency components of the input signal will be blocked, not passed to the output. With the four knobs, we can adjust how much of the *power* of the input signal at the frequency corresponding to the knob is passed to the output. The knobs work in decibels: putting the first knob to 0 dB means that the input power at 50 Hz is passed $1:1$ to the output ($P_y|_{f=50} = P_x|_{f=50}$); putting it at -10 dB means that only 10% of the input power at 50 Hz is passed to the output ($P_y|_{f=50} = 0.10\ P_x|_{f=50}$), etc.

Note: the equalizer settings in the above illustration are its default settings. Your task is to calculate modified equalizer knob settings that result in the desired output signal $y(t)$.

It is given that:

- The power of the output signal is 20 W,
- The equalizer gain at 100 Hz is set at 0 dB, and
- The equalizer gain at 50 Hz is set 10 dB higher than the equalizer gain at 500 Hz.

Which of the following statements is correct?

A. The equalizer gain at 500 Hz is set at -2.33 dB.

B. The equalizer gain at 500 Hz is set at -10.11 dB.

C. The equalizer gain at 50 Hz is set at 1.93 dB.

D. None of the other statements is correct.

Solutions

S|2.1 Signal models

S.2-1

Using $\sin^2 u = \frac{1}{2}(1-\cos(2u))$, (A.5), we obtain:

$$x(t) = 2 - 1.5\sin^2\left(15\pi t + \frac{\pi}{6}\right) = 2 - 1.5 \cdot \frac{1}{2}\left(1 - \cos\left(30\pi t + \frac{\pi}{3}\right)\right)$$
$$= 2 - 0.75 + 0.75\cos\left(30\pi t + \frac{\pi}{3}\right) = 1.25 + 0.75\cos\left(30\pi t + \frac{\pi}{3}\right)$$

The dynamic range D of this signal is 1.5 V, as the signal oscillates between $+0.5$ and $+2$ V. Dividing D by the required step size yields $\frac{1.5}{0.05} = 30$ signal levels. With $p = 5$ bits we get $Q = 2^p = 32$ possibilities, which is just sufficient, as it leads to a step size Δx of $\frac{1.5}{32} = 0.0469$ V. Taking $p = 4$ bits would yield a step size of $\Delta x = \frac{1.5}{16} = 0.0938$ V, which is too large. The number of bits p required to fulfill the step size requirement of 0.05 V is 5.

S.2-2

The dynamic range D of the signal is 5 V. With 10 bits, the quantization step size equals $\Delta x = \frac{D}{2^p} = \frac{5}{2^{10}} = 4.88 \cdot 10^{-3}$ V. The maximum quantization error equals half the step size Δx. Hence, the maximum quantization error is $2.44 \cdot 10^{-3}$ V.

S.2-3

(a) $x_a(t) = 4\cos(2\pi t + \pi)$: The phase (= relative phase = phase shift) θ, is π rad. The time shift $t_0 = \frac{\theta}{2\pi f_0} = \frac{\pi}{2\pi} = \frac{1}{2}$ s; see (2.3).

(b) $x_b(t) = \sin(5\pi t) = \cos\left(5\pi t - \frac{\pi}{2}\right)$: $\theta = -\frac{\pi}{2}$ rad, $t_0 = \frac{-\frac{\pi}{2}}{5\pi} = -\frac{1}{10}$ s

(c) $x_c(t) = \pi\cos(36\pi t + 12)$: $\theta = 12$ rad, $t_0 = \frac{1}{3\pi}$ s

(d) $x_d(t) = \sin(2\pi t + 2) = \cos\left(2\pi t + 2 - \frac{\pi}{2}\right)$: $\theta = 2 - \frac{\pi}{2}$ rad, $t_0 = \frac{4-\pi}{4\pi}$ s

(e) $x_e(t) = -\frac{3}{2}\cos(\pi t) = \frac{3}{2}\cos(\pi t \pm \pi)$: $\theta = \pm\pi$ rad, $t_0 = \pm 1$ s

(f) $x_f(t) = 2\cos^2\left(2\pi t + \frac{\pi}{6}\right) = 1 + \cos\left(4\pi t + \frac{\pi}{3}\right)$: $\theta = \frac{\pi}{3}$ rad, $t_0 = \frac{1}{12}$ s

where we used (A.4).

Note that in (e) we obtain an ambiguity, the phase can be $\pm\pi$ and the corresponding time shift ± 1 s. Without more information, we cannot tell whether this signal is either lagging behind, or is ahead of the zero-phase cosine.

Note that in drawing the spectrum of a signal, the amplitude must be positive, and the phase must lie in the range $[-\pi, \pi]$, see Section 2.3.4. After some operations, all signals in this exercise now fulfill these two requirements, except signal $x_c(t)$ which has a phase $\theta = 12$ rad. In drawing its spectrum, its phase must be wrapped into the range $[-\pi, \pi]$ by adding or subtracting integer multiples of 2π. Here, the 12 rad ($\approx 3.82\pi$ rad) is brought into the range by subtracting 4π rad, resulting in a phase of $x_c(t)$ of $12 - 4\pi$ rad (≈ -0.57 rad). The time shift does *not* change: only when drawing the signal spectrum the phase is wrapped into the range $[-\pi, \pi]$. In time shift calculations we work with the unwrapped phase.

S.2-4

The ratio of the two frequencies equals $\frac{2.4}{8.4}$, which when written using (the smallest possible) integers equals $\frac{2}{7}$, a rational number. Then, $f_1 = n_1 f_0 = 2 f_0 = 2.4$ Hz and $f_2 = n_2 f_0 = 7 f_0 = 8.4$ Hz, so $f_0 = 1.2$ Hz. Hence, the sum of the two sinusoidal components will be periodic, with fundamental frequency $f_0 = 1.2$ Hz and period $T_0 = \frac{1}{1.2} = \frac{5}{6} \approx 0.8333$ s.

S.2-5

(a) $x_a(t) = \sin(20\pi t) = \sin\big(10(2\pi)t\big)$: $\quad f_0 = 10$ Hz, $\quad T_0 = \frac{1}{10}$ s

(b) $x_b(t) = \cos(30\pi t) = \cos\big(15(2\pi)t\big)$: $\quad f_0 = 15$ Hz, $\quad T_0 = \frac{1}{15}$ s

(c) $x_c(t) = \cos(36\pi t) = \cos\big(18(2\pi)t\big)$: $\quad f_0 = 18$ Hz, $\quad T_0 = \frac{1}{18}$ s

(d) $x_d(t) = x_a(t) + x_b(t) = \sin(20\pi t) + \cos(30\pi t) = \sin\big(10(2\pi)t\big) + \cos\big(15(2\pi)t\big)$
Find the common frequency: $x_d(t) = \sin\big(2(5)(2\pi)t\big) + \cos\big(3(5)(2\pi)t\big)$
Hence: $\quad f_0 = 5$ Hz, $\quad T_0 = \frac{1}{5}$ s

(e) $x_e(t) = x_a(t) + x_c(t) = \sin(20\pi t) + \cos(36\pi t) = \sin\big(10(2\pi)t\big) + \cos\big(18(2\pi)t\big)$
Find the common frequency: $x_e(t) = \sin\big(5(2)(2\pi)t\big) + \cos\big(9(2)(2\pi)t\big)$
Hence: $\quad f_0 = 2$ Hz, $\quad T_0 = \frac{1}{2}$ s

(f) $x_f(t) = x_b(t) + x_c(t) = \cos(30\pi t) + \cos(36\pi t) = \cos\big(15(2\pi)t\big) + \cos\big(18(2\pi)t\big)$
Find the common frequency: $x_f(t) = \sin\big(5(3)(2\pi)t\big) + \cos\big(6(3)(2\pi)t\big)$
Hence: $\quad f_0 = 3$ Hz, $\quad T_0 = \frac{1}{3}$ s

S.2-6

(a) $x_a(t) = 4\cos\left(5\pi t - \frac{\pi}{3}\right) = 4\cos\left(2.5(2\pi)t - \frac{\pi}{3}\right)$: $\quad f_0 = 2.5$ Hz, $\quad T_0 = 0.4$ s

(b) $x_b(t) = 3\cos\left(9\pi t + \frac{\pi}{6}\right) = 3\cos\left(4.5(2\pi)t + \frac{\pi}{6}\right)$: $\quad f_0 = 4.5$ Hz, $\quad T_0 = \frac{2}{9} \approx 0.2222$ s

(c) $x_c(t) = 5\sin\left(11\pi t - \frac{\pi}{4}\right) = 5\sin\left(5.5(2\pi)t - \frac{\pi}{4}\right)$: $\quad f_0 = 5.5$ Hz, $\quad T_0 = \frac{2}{11} \approx 0.1818$ s

(d) $$x_d(t) = x_a(t) + x_b(t) = 4\cos\left(5\pi t - \frac{\pi}{3}\right) + 3\cos\left(9\pi t + \frac{\pi}{6}\right)$$
$$= 4\cos\left(5(0.5)(2\pi)t - \frac{\pi}{3}\right) + 3\cos\left(9(0.5)(2\pi)t + \frac{\pi}{6}\right)$$
Hence: $\quad f_0 = 0.5$ Hz, $\quad T_0 = 2$ s

(e) $$x_e(t) = x_a(t) + x_c(t) = 4\cos\left(5\pi t - \frac{\pi}{3}\right) + 5\sin\left(11\pi t - \frac{\pi}{4}\right)$$
$$= 4\cos\left(5(0.5)(2\pi)t - \frac{\pi}{3}\right) + 5\sin\left(11(0.5)(2\pi)t - \frac{\pi}{4}\right)$$
Hence: $\quad f_0 = 0.5$ Hz, $\quad T_0 = 2$ s

(f) $$x_f(t) = x_b(t) + x_c(t) = 3\cos\left(9\pi t + \frac{\pi}{6}\right) + 5\sin\left(11\pi t - \frac{\pi}{4}\right)$$
$$= 3\cos\left(9(0.5)(2\pi)t + \frac{\pi}{6}\right) + 5\sin\left(11(0.5)(2\pi)t - \frac{\pi}{4}\right)$$
Hence: $\quad f_0 = 0.5$ Hz, $\quad T_0 = 2$ s

Note that the phase does not matter, and that we are always looking for the *largest* frequency that occurs an *integer* number of times.

S.2-7

(a) $$\begin{aligned} x_a(t) &= 2\sin\left(\tfrac{3\pi}{4}t + \tfrac{\pi}{4}\right) + 7\sin\left(4\pi t - \tfrac{\pi}{6}\right) \\ &= 2\sin\left(\tfrac{3}{8}(2\pi)t + \tfrac{\pi}{4}\right) + 7\sin\left(2(2\pi)t - \tfrac{\pi}{6}\right) \\ &= 2\sin\left(3(\tfrac{1}{8})(2\pi)t + \tfrac{\pi}{4}\right) + 7\sin\left(16(\tfrac{1}{8})(2\pi)t - \tfrac{\pi}{6}\right) \end{aligned}$$

Hence: signal $x_a(t)$ is periodic with $f_0 = \frac{1}{8} = 0.125$ Hz, $T_0 = 8$ s

(b) Using $\sin^2 u = \frac{1}{2}(1-\cos(2u))$, (A.5), $\sin u = \cos\left(u - \frac{\pi}{2}\right)$ and $\cos(u \pm \pi) = -\cos u$:

$$\begin{aligned} x_b(t) &= -\tfrac{1}{2} + \sin^2(2\pi t) - \tfrac{1}{2}\sin\left(4\pi t - \tfrac{\pi}{2}\right) \\ &= -\tfrac{1}{2} + \tfrac{1}{2}(1 - \cos(4\pi t)) - \tfrac{1}{2}\cos\left(4\pi t - \tfrac{\pi}{2} - \tfrac{\pi}{2}\right) \\ &= -\tfrac{1}{2} + \tfrac{1}{2} - \tfrac{1}{2}\cos(4\pi t) + \tfrac{1}{2}\cos(4\pi t) = 0 \end{aligned}$$

Hence: signal $x_b(t)$ is not periodic.

(c) Using $\sin u = \cos\left(u - \frac{\pi}{2}\right)$ and $\cos(u \pm \pi) = -\cos u$:

$$\begin{aligned} x_c(t) &= -3 + \cos(15\pi t) + \sin\left(15\pi t - \tfrac{\pi}{2}\right) + 4\cos\left(30\pi t + \tfrac{\pi}{16}\right) - 5\sin\left(20\pi t + \tfrac{\pi}{3}\right) \\ &= -3 + \cos(15\pi t) + \cos(15\pi t - \pi) + 4\cos\left(30\pi t + \tfrac{\pi}{16}\right) - 5\sin\left(20\pi t + \tfrac{\pi}{3}\right) \\ &= -3 + \cos(15\pi t) - \cos(15\pi t) + 4\cos\left(30\pi t + \tfrac{\pi}{16}\right) - 5\sin\left(20\pi t + \tfrac{\pi}{3}\right) \\ &= -3 + 4\cos\left(30\pi t + \tfrac{\pi}{16}\right) - 5\sin\left(20\pi t + \tfrac{\pi}{3}\right) \\ &= -3 + 4\cos\left(3(5)(2\pi)t + \tfrac{\pi}{16}\right) - 5\sin\left(2(5)(2\pi)t + \tfrac{\pi}{3}\right) \end{aligned}$$

Hence: signal $x_c(t)$ is periodic with $f_0 = 5$ Hz, $T_0 = 0.2$ s

S.2-8

In this exercise, we use the sifting property (B.12) of the Dirac delta function.

(a) Substitute $\delta(t-5) = 0$ for all values of t except $t = 5$ s:

$$\int_{-\infty}^{\infty} e^{-\alpha t^2}\,\delta(t-5)\,\mathrm{d}t = e^{-\alpha 5^2}\cdot 1 = e^{-25\alpha}$$

where we used the fact that the integral of a Dirac delta function equals one (when the Dirac delta function occurs within the integral limits, of course).

(b) Note that $\delta(t+5) = 0$ at all values of t except at $t = -5$ s, which is outside the integral limits. Hence, the integral is 0.

(c) $$\int_{-\infty}^{\infty} \left(3 + \cos(3\pi t)\right)\delta(t)\,\mathrm{d}t = 3 + \cos(3\pi 0) = 3 + 1 = 4$$

S.2-9

In this exercise, we use the sifting property (B.12) of the Dirac delta function.

(a) Substitute $\delta(t-4) = 0$ for all values of t except $t = 4$ s:

$$\int_0^6 \cos(4\pi t)\,\delta(t-4)\,\mathrm{d}t = \cos(4\pi(4))\cdot 1 = 1$$

(b) Note that $\delta(t-4) = 0$ at all values of t except at $t = 4$ s, which is outside the integral limits. Hence, the integral is 0.

(c) $$\int_0^6 \cos(4\pi t)\,\delta(t-0.75)\,\mathrm{d}t = \cos(4\pi(0.75))\cdot 1 = -1$$

(d) $\int_{-\infty}^{\infty} (t-4)^2 \delta(t-4)\,\mathrm{d}t = (4-4)^2 \cdot 1 = 0$

(e) $\int_{-\infty}^{\infty} t^2 \delta(t-4)\,\mathrm{d}t = 4^2 \cdot 1 = 16$

(f) $\int_{-\infty}^{\infty} t^4 \delta(t-4)\,\mathrm{d}t = 4^4 \cdot 1 = 256$

S | 2.2 Signal symmetry

S.2-10

(a) The average of a periodic signal can be computed by considering one period: integrate the signal over that period, then divide by the length of the period:
AVG $= \frac{1}{4}\int_0^4 x_a(t)\,\mathrm{d}t = \frac{1}{4}\Big(0 + 2\int_1^2 (2-t)\,\mathrm{d}t + 0\Big) = \frac{1}{2}\left[2t - \frac{1}{2}t^2\right]_1^2 = \frac{1}{2}\frac{1}{2} = \frac{1}{4}$
Signal $x_a(t)$ is even, not half-wave odd.

(b) Signal $x_b(t)$ is a sinusoid which oscillates about 0, hence the average is 0; no need to do any calculations. It is neither even nor odd; it is half-wave odd.

(c) Signal $x_c(t)$ is a sinusoid which oscillates about 1; the average is 1. It has no symmetry properties.

(d) Signal $x_d(t)$ is a signal which oscillates about 0, in a symmetric fashion, therefore the average is 0. It is odd, not half-wave odd.

(e) Signal $x_e(t)$ is a square wave which oscillates about 0.5; half of the time it is 1, half of the time it is 0: the average is 0.5. It is even, not half-wave odd.

(f) Signal $x_f(t)$ is a triangular wave which oscillates about 0; the average is 0. It is even and half-wave odd: the signal has quarter-wave symmetry.

S.2-11

(a) Rephrasing the definition of a half-wave odd signal on page 18 of the companion book, Theory, for a half-wave even signal, if it exists, would lead to the following. Imagine taking a sample of a half-wave even signal at any arbitrary moment in time t', and then taking another sample half a period later (or earlier), at $t' \pm \frac{T_0}{2}$; this yields the same signal value *and with the same* sign. That is:

$$x(t \pm \tfrac{T_0}{2}) = x(t)$$

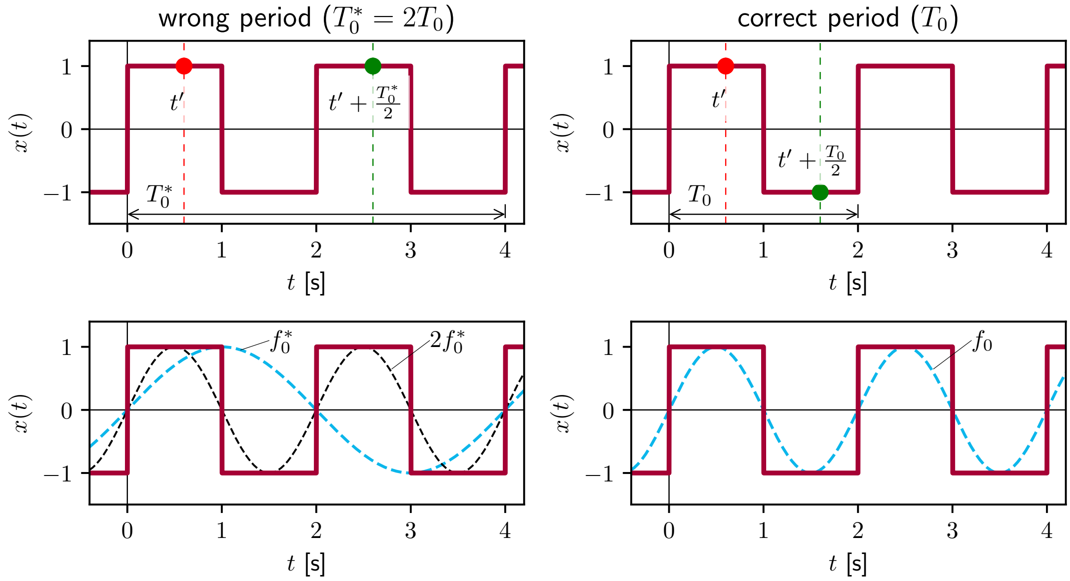
We just formulated an expression similar to condition number 3 for a half-wave even signal. In this definition, we use T_0, which implies that for a signal to be half-wave even it must be periodic. This condition is the same as condition number 1 for the half-wave odd signal. In contrast to the half-wave odd signal, the expression above also holds for signals which have a non-zero average (to test, simply add a non-zero constant c to the signal). We conclude that a signal is defined to have half-wave even symmetry, if this symmetry property exists, when two conditions are met:

1. The signal is periodic (period T_0).
2. The following property holds: $x(t \pm \frac{T_0}{2}) = x(t)$.

(b) When a signal value and sign at any time t is identical to that signal's value and sign half a period later (or earlier), one has determined the *wrong period* of the signal! Then 'half a period' *is* the period. The half-wave even symmetry property is a common and persistent misconception: this symmetry property does not exist.

Note that in Exercise *e*.1-2(c) the rectified sine wave had a period T_0 of $\frac{1}{2}$ s, while the signal model used period 1 s; this causes all odd-indexed building blocks to be zero (see Solution *s*.1-2(c)). It is the consequence of setting the *wrong* period in your signal model (namely 1 s).

In the figure below we further illustrate this point, using a periodic square wave, which is odd and half-wave odd, as an example.



In the left column, we deliberately assumed the wrong period, $T_0^* = 4$ s, which is twice the actual period $T_0 = 2$ s. Within T_0^* the signal repeats itself two times. With this *wrong* period, we *do* see half-wave even symmetry, as $x(t \pm \frac{T_0^*}{2}) = x(t)$ (top, the red and green dots). As a result of taking the wrong period, the fundamental frequency of all signal model building blocks, f_0^*, is half what it should be. The first building block, with $1f_0^*$, does not represent the signal very well (bottom, blue dashed curve) and would be useless in building this signal, its frequency is too low. In contrast, the second building block, with frequency $2f_0^*$, *does* make a sensible contribution to modeling this signal (black dashed curve). With the period T_0^* in our signal model, all odd-indexed building blocks in that model are zero.
The right column shows the results when assuming the correct period T_0, the signal is half-wave odd, $x(t \pm \frac{T_0}{2}) = -x(t)$ (top, the red and green dots). Here, the first building block (bottom, blue dashed curve), with frequency f_0, helps to construct the signal.

s|**2.3** Sinusoid building block

In this exercise, we use the fact that the magnitude and argument of a complex number $X = a + jb$, with real part $a = \mathrm{Re}(X)$ and imaginary part $b = \mathrm{Im}(X)$, are given by: *s*.2-12

$|X| = \sqrt{\mathrm{Re}^2(X) + \mathrm{Im}^2(X)}$ and $\angle X = \arctan\left(\frac{\mathrm{Im}(X)}{\mathrm{Re}(X)}\right)$

We also use that the real and imaginary parts of X are given by:

$\mathrm{Re}(X) = |X| \cos \angle X$ and $\mathrm{Im}(X) = |X| \sin \angle X$ (see Appendix C)

(a) $A = 4 + j4$: magnitude $|A| = \sqrt{4^2 + 4^2} = \sqrt{32} = 4\sqrt{2}$ argument $\angle A = \arctan\left(\frac{4}{4}\right) = \frac{\pi}{4}$

$A = 4\sqrt{2}e^{j\frac{\pi}{4}}$

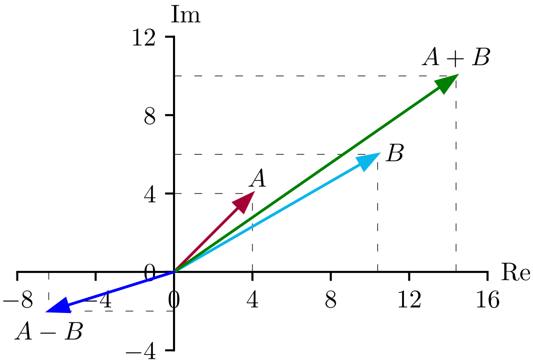
$B = 12e^{j\frac{\pi}{6}}$: magnitude $|B| = 12$ argument $\angle B = \frac{\pi}{6}$

$\mathrm{Re}(B) = 12\cos\left(\frac{\pi}{6}\right) = 12 \cdot \frac{\sqrt{3}}{2} = 6\sqrt{3}$ $\mathrm{Im}(B) = 12\sin\left(\frac{\pi}{6}\right) = 12 \cdot \frac{1}{2} = 6$

$B = 6\sqrt{3} + j6$

(b) $A + B = (4 + 6\sqrt{3}) + j10$

(c) $A - B = (4 - 6\sqrt{3}) - j2$



(d) Cartesian form:

$$AB = (4 + j4)\,(6\sqrt{3} + j6)$$
$$= 24(\sqrt{3} - 1) + j24(\sqrt{3} + 1)$$

Polar form:

$$AB = 4\sqrt{2}e^{j\frac{\pi}{4}}12e^{j\frac{\pi}{6}} = 48\sqrt{2}e^{j\frac{5\pi}{12}}$$

To show that both answers are equivalent:

$$\mathrm{Re}(AB) = 48\sqrt{2}\cos\left(\frac{5\pi}{12}\right) = 48\sqrt{2}\cos\left(\frac{\pi}{4} + \frac{\pi}{6}\right)$$
$$= 48\sqrt{2}\left(\cos\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{6}\right) - \sin\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{6}\right)\right) = 48\sqrt{2}\left(\frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\frac{1}{2}\right) = 24(\sqrt{3} - 1)$$
$$\mathrm{Im}(AB) = 48\sqrt{2}\sin\left(\frac{5\pi}{12}\right) = 48\sqrt{2}\sin\left(\frac{\pi}{4} + \frac{\pi}{6}\right)$$
$$= 48\sqrt{2}\left(\sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{6}\right) + \cos\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{6}\right)\right) = 48\sqrt{2}\left(\frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\frac{1}{2}\right) = 24(\sqrt{3} + 1)$$

where we used (A.7) and (A.6), respectively, to compute $\mathrm{Re}(AB)$ and $\mathrm{Im}(AB)$.

Or check: $|AB| = \sqrt{\mathrm{Re}^2(AB) + \mathrm{Im}^2(AB)} = \sqrt{576(\sqrt{3} - 1)^2 + 576(\sqrt{3} + 1)^2} = 48\sqrt{2}$

$$\angle AB = \arctan\left(\frac{\mathrm{Im}(AB)}{\mathrm{Re}(AB)}\right) = \arctan\left(\frac{24(\sqrt{3} + 1)}{24(\sqrt{3} - 1)}\right) = \frac{5\pi}{12}$$

(e) Cartesian form: $\frac{A}{B} = \frac{4 + j4}{6\sqrt{3} + j6} = \frac{4 + j4}{6\sqrt{3} + j6}\frac{6\sqrt{3} - j6}{6\sqrt{3} - j6} = \frac{1}{6}(\sqrt{3} + 1) + j\frac{1}{6}(\sqrt{3} - 1)$

Polar form: $\frac{A}{B} = \frac{4\sqrt{2}}{12}\frac{e^{j\frac{\pi}{4}}}{e^{j\frac{\pi}{6}}} = \frac{1}{3}\sqrt{2}e^{j\frac{\pi}{12}}$

Check: $\left|\frac{A}{B}\right| = \sqrt{\mathrm{Re}^2\left(\frac{A}{B}\right) + \mathrm{Im}^2\left(\frac{A}{B}\right)} = \sqrt{\frac{1}{36}(\sqrt{3} + 1)^2 + \frac{1}{36}(\sqrt{3} - 1)^2} = \frac{1}{3}\sqrt{2}$

$$\angle\frac{A}{B} = \arctan\left(\frac{\mathrm{Im}\left(\frac{A}{B}\right)}{\mathrm{Re}\left(\frac{A}{B}\right)}\right) = \arctan\left(\frac{\frac{1}{6}(\sqrt{3} - 1)}{\frac{1}{6}(\sqrt{3} + 1)}\right) = \frac{\pi}{12}$$

Or check: $\mathrm{Re}\left(\frac{A}{B}\right) = \mathrm{Re}\left(\frac{1}{3}\sqrt{2}e^{j\frac{\pi}{12}}\right) = \frac{1}{3}\sqrt{2}\cos\left(\frac{\pi}{12}\right) = \frac{1}{6}(\sqrt{3} + 1)$

$$\mathrm{Im}\left(\frac{A}{B}\right) = \mathrm{Im}\left(\frac{1}{3}\sqrt{2}e^{j\frac{\pi}{12}}\right) = \frac{1}{3}\sqrt{2}\sin\left(\frac{\pi}{12}\right) = \frac{1}{6}(\sqrt{3} - 1)$$

S.2-13 Compute the Taylor series of e^{jx} (with j the imaginary unit for which $j^2 = -1$):

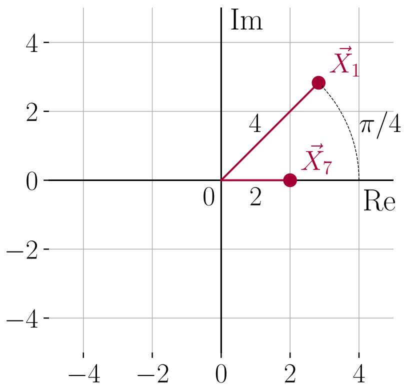
$$e^{jx} = \sum_{k=0}^{k=\infty}\frac{(jx)^k}{k!} = 1 + jx - \frac{x^2}{2!} - j\frac{x^3}{3!} + \frac{x^4}{4!} + j\frac{x^5}{5!} - \frac{x^6}{6!} - j\frac{x^7}{7!} + \ldots$$
$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\right) + j\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\right)$$
$$= \cos x + j\sin x$$

qed

S.2-14

(a) Signal $x_a(t) = 4\cos\left(2\pi t + \frac{\pi}{4}\right) - 2\sin\left(14\pi t - \frac{\pi}{2}\right)$ has two components, with frequencies 1 and 7 Hz, respectively. The fundamental frequency f_0 is then 1 Hz.
Using $\sin u = \cos\left(u - \frac{\pi}{2}\right)$ and $\cos(u \pm \pi) = -\cos u$:

$$\begin{aligned}
x_a(t) &= 4\cos\left(2\pi t + \tfrac{\pi}{4}\right) - 2\cos\left(14\pi t - \tfrac{\pi}{2} - \tfrac{\pi}{2}\right) \\
&= 4\cos\left(2\pi t + \tfrac{\pi}{4}\right) - 2\cos(14\pi t - \pi) \\
&= 4\cos\left(2\pi t + \tfrac{\pi}{4}\right) + 2\cos(14\pi t) \\
&= \mathrm{Re}\big(\underbrace{4e^{j(\frac{\pi}{4})}}_{\text{phasor } \vec{X}_1} e^{j2\pi t}\big) + \mathrm{Re}\big(\underbrace{2e^{j(0)}}_{\text{phasor } \vec{X}_7} e^{j14\pi t}\big)
\end{aligned}$$

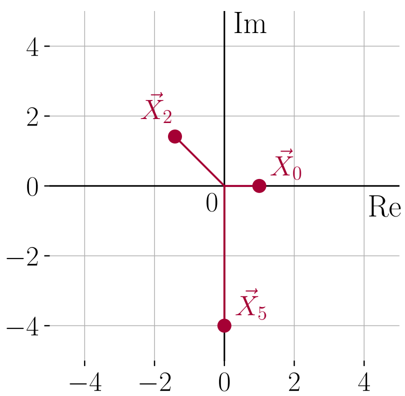


phasor $\vec{X}_1 = 4\left(\frac{1}{2}\sqrt{2} + j\frac{1}{2}\sqrt{2}\right) = 2\sqrt{2}(1+j)$
phasor $\vec{X}_7 = 2$
Both phasors are drawn on the right.
The magnitude and phase of the phasors $\vec{X}_1$ and $\vec{X}_7$ immediately follow from the description of $x_a(t)$ as the real part of the complex exponential function. That is:
$|\vec{X}_1| = 4, \quad \angle\vec{X}_1 = \theta_1 = \frac{\pi}{4}$
$|\vec{X}_7| = 2, \quad \theta_7 = 0$

(b) Signal $x_b(t) = 1 - 2\sin\left(2\pi t + \frac{\pi}{4}\right) + 4\cos\left(5\pi t - \frac{\pi}{2}\right)$ has a fundamental frequency f_0 of 0.5 Hz. The constant '1' is considered a 'cosine with zero frequency', its phasor is then $\vec{X}_0$. Given that the frequencies of the other two components (1 and 2.5 Hz) equal, respectively, 2 and 5 times the fundamental frequency, we get phasors $\vec{X}_2$ and $\vec{X}_5$.
Using $\sin u = \cos\left(u - \frac{\pi}{2}\right)$, $\cos(u \pm \pi) = -\cos u$, and keeping the phase within $[-\pi, \pi]$:

$$\begin{aligned}
x_b(t) &= 1\cos(0\pi t) - 2\cos\left(2\pi t + \tfrac{\pi}{4} - \tfrac{\pi}{2}\right) + 4\cos\left(5\pi t - \tfrac{\pi}{2}\right) \\
&= 1\cos(0\pi t) - 2\cos\left(2\pi t - \tfrac{\pi}{4}\right) + 4\cos\left(5\pi t - \tfrac{\pi}{2}\right) \\
&= 1\cos(0\pi t + 0) + 2\cos\left(2\pi t + \tfrac{3\pi}{4}\right) + 4\cos\left(5\pi t - \tfrac{\pi}{2}\right) \\
&= \mathrm{Re}\big(\underbrace{1e^{j(0)}}_{\text{phasor } \vec{X}_0} e^{j0t}\big) + \mathrm{Re}\big(\underbrace{2e^{j(\frac{3\pi}{4})}}_{\text{phasor } \vec{X}_2} e^{j2\pi t}\big) + \mathrm{Re}\big(\underbrace{4e^{j(-\frac{\pi}{2})}}_{\text{phasor } \vec{X}_5} e^{j5\pi t}\big)
\end{aligned}$$

phasor $\vec{X}_0 = 1$
phasor $\vec{X}_2 = 2\left(-\frac{1}{2}\sqrt{2} + j\frac{1}{2}\sqrt{2}\right) = -\sqrt{2}(1-j)$
phasor $\vec{X}_5 = -4j$
All phasors are drawn on the right.
Magnitudes and phases:
$|\vec{X}_0| = 1, \quad \theta_0 = 0$
$|\vec{X}_2| = 2, \quad \theta_2 = \frac{3\pi}{4}$
$|\vec{X}_5| = 4, \quad \theta_5 = -\frac{\pi}{2}$



(c) Signal $x_c(t) = -2 + 4\sin(4\pi t) + 3\cos\left(4\pi t + \frac{\pi}{4}\right)$ has a fundamental frequency f_0 of 2 Hz; the sine and cosine components have the same frequency. We first compute the phasors belonging to these components and then add them together to obtain phasor $\vec{X}_1$.

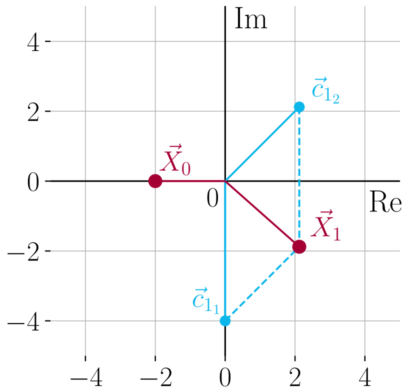
Using $\sin u = \cos\left(u - \frac{\pi}{2}\right)$ and $\cos(u \pm \pi) = -\cos u$:

$$\begin{aligned}
x_c(t) &= -2\cos(0\pi t) + 4\cos\left(4\pi t - \frac{\pi}{2}\right) + 3\cos\left(4\pi t + \frac{\pi}{4}\right)\\
&= 2\cos(0\pi t + \pi) + 4\cos\left(4\pi t - \frac{\pi}{2}\right) + 3\cos\left(4\pi t + \frac{\pi}{4}\right)\\
&= \mathrm{Re}\left(2e^{j(\pi)}e^{j0t}\right) + \mathrm{Re}\left(\underbrace{4e^{j(-\frac{\pi}{2})}}_{\text{component } \vec{c}_{1_1}} e^{j4\pi t}\right) + \mathrm{Re}\left(\underbrace{3e^{j(\frac{\pi}{4})}}_{\text{component } \vec{c}_{1_2}} e^{j4\pi t}\right)\\
&= \mathrm{Re}\left(\underbrace{2e^{j(\pi)}}_{\text{phasor } \vec{X}_0} e^{j0t}\right) + \mathrm{Re}\left(\underbrace{\left(4e^{j(-\frac{\pi}{2})} + 3e^{j(\frac{\pi}{4})}\right)}_{\text{phasor } \vec{X}_1} e^{j4\pi t}\right)
\end{aligned}$$

phasor $\vec{X}_0 = -2$

phasor $\vec{X}_1 = \vec{c}_{1_1} + \vec{c}_{1_2} = (0 - 4j) + \left(\frac{3}{2}\sqrt{2} + j\frac{3}{2}\sqrt{2}\right)$

$$= \frac{3}{2}\sqrt{2} + j\left(\frac{3}{2}\sqrt{2} - 4\right)$$



Both phasors are drawn on the right, with the individual components of $\vec{X}_1$ in blue.

Magnitudes and phases:

$|\vec{X}_0| = 2, \quad \theta_0 = \pi$

$$|\vec{X}_1| = \sqrt{\left(\frac{3}{2}\sqrt{2}\right)^2 + \left(\frac{3}{2}\sqrt{2} - 4\right)^2} = \sqrt{25 - 12\sqrt{2}}$$

$$\theta_1 = \arctan\left(\frac{\frac{3}{2}\sqrt{2} - 4}{\frac{3}{2}\sqrt{2}}\right) = \arctan\left(-\frac{4}{3}\sqrt{2} + 1\right)$$

Here, we used the expressions for the modulus $|X| = \sqrt{a^2 + b^2}$ and phase $\theta = \arctan\left(\frac{b}{a}\right)$ of a complex number $X = a + jb$ given in Section C.1.

Note that using the phasors $\vec{X}_k$ the single-sided amplitude and phase spectra of periodic signal $x(t)$ can be directly obtained. Plotting, respectively, the phasor magnitude $|\vec{X}_k|$ and phase angle $\angle\vec{X}_k$ as a function of kf_0, with $k \in \mathbb{N}$ and f_0 the signal's fundamental frequency in [Hz], yields the single-sided amplitude and phase spectra.

Note that periodic signal $x(t)$ can be written as the real part of the sum of all (here K) phasors $\vec{X}_k$ multiplied by the complex exponential basis function with frequency kf_0:

$$x(t) = \mathrm{Re}\left(\sum_{k=0}^{K-1} \vec{X}_k e^{j2\pi k f_0 t}\right)$$

s.2-15

Hint: of course, when you have obtained the phasors $\vec{X}_k$ in Exercise *e*.2-14, there is no need to do any calculations again. For the sake of completeness, we do include these below.

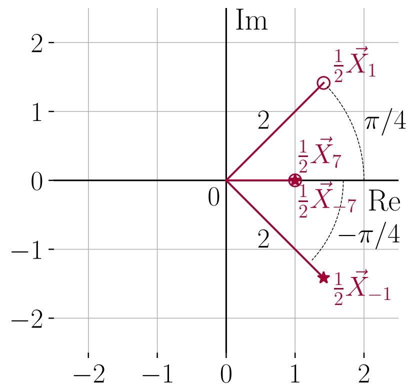
(a) Signal $x_a(t)$ (fundamental frequency $f_0 = 1$ Hz) can be written as:

$$x_a(t) = \underbrace{2e^{j(\frac{\pi}{4})}}_{\frac{1}{2}\vec{X}_1} e^{j2\pi t} + \underbrace{2e^{j(-\frac{\pi}{4})}}_{\frac{1}{2}\vec{X}_{-1}} e^{-j2\pi t} + \underbrace{1e^{j(0)}}_{\frac{1}{2}\vec{X}_7} e^{j14\pi t} + \underbrace{1e^{j(0)}}_{\frac{1}{2}\vec{X}_{-7}} e^{-j14\pi t}$$

$$\tfrac{1}{2}\vec{X}_1 \qquad\quad = 2\left(\tfrac{1}{2}\sqrt{2} + j\tfrac{1}{2}\sqrt{2}\right) = \sqrt{2}(1+j)$$

$$\tfrac{1}{2}\vec{X}_{-1} = \tfrac{1}{2}\vec{X}_1^* = 2\left(\tfrac{1}{2}\sqrt{2} - j\tfrac{1}{2}\sqrt{2}\right) = \sqrt{2}(1-j)$$

$$\tfrac{1}{2}\vec{X}_7 \;= \tfrac{1}{2}\vec{X}_{-7} = 1$$



All complex conjugate phasor pairs are drawn on the right. Note the difference in the axes limits with respect to the figures showing the 'single-sided' phasors in Solution *s*.2-14. That is because we show $\frac{1}{2}\vec{X}_k$ and $\frac{1}{2}\vec{X}_{-k}$.

(b) Signal $x_b(t)$ (fundamental frequency $f_0 = 0.5$ Hz) can be written as:

$$x_b(t) = \underbrace{1e^{j(0)}}_{\vec{X}_0}\, e^{j0t} + \underbrace{1e^{j(\frac{3\pi}{4})}}_{\frac{1}{2}\vec{X}_2}\, e^{j2\pi t} + \underbrace{1e^{j(-\frac{3\pi}{4})}}_{\frac{1}{2}\vec{X}_{-2}}\, e^{-j2\pi t} + \underbrace{2e^{j(-\frac{\pi}{2})}}_{\frac{1}{2}\vec{X}_5}\, e^{j5\pi t} + \underbrace{2e^{j(\frac{\pi}{2})}}_{\frac{1}{2}\vec{X}_{-5}}\, e^{-j5\pi t}$$

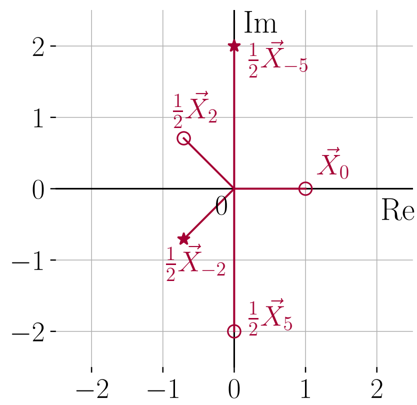
$$\tfrac{1}{2}\vec{X}_0 \qquad\quad = 1$$

$$\tfrac{1}{2}\vec{X}_2 \qquad\quad = -\tfrac{1}{2}\sqrt{2}(1-j)$$

$$\tfrac{1}{2}\vec{X}_{-2} = \tfrac{1}{2}\vec{X}_2^* = -\tfrac{1}{2}\sqrt{2}(1+j)$$

$$\tfrac{1}{2}\vec{X}_5 \qquad\quad = -2j$$

$$\tfrac{1}{2}\vec{X}_{-5} = \tfrac{1}{2}\vec{X}_5^* = 2j$$



All complex conjugate phasor pairs are drawn on the right. At the zero frequency ($k = 0$) the phasor is always real-valued and indicated by $\vec{X}_0$.

We can calculate the 'total' complex-valued function $c_+(t)$ corresponding to the contributions of all positive frequencies to building the signal $x_b(t)$ as follows:

$$c_+(t) = \tfrac{1}{2}\vec{X}_0 + \tfrac{1}{2}\vec{X}_2 e^{j2\pi(2)(0.5)t} + \tfrac{1}{2}\vec{X}_5 e^{j2\pi(5)(0.5)t}$$

Similarly, for the 'total' complex-valued function $c_-(t)$ corresponding to the contributions of the negative frequencies:

$$c_-(t) = \tfrac{1}{2}\vec{X}_0 + \tfrac{1}{2}\vec{X}_{-2} e^{-j2\pi(2)(0.5)t} + \tfrac{1}{2}\vec{X}_{-5} e^{-j2\pi(5)(0.5)t} = c_+^*(t)$$

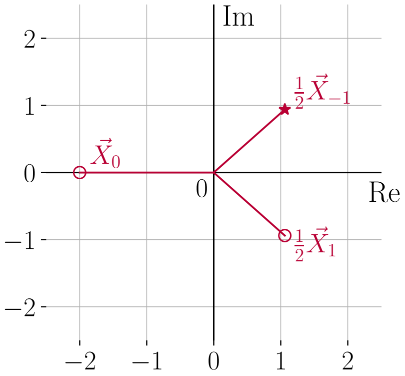
Both complex functions get *half* of the signal average, corresponding to $\frac{1}{2}\vec{X}_0 = \frac{1}{2}X_0$. This is because when *adding* these functions the total average should correspond to the value of $\vec{X}_0 = X_0$.
Signal $x_b(t)$ can be described as: $x_b(t) = c_+(t) + c_-(t) = c_+(t) + c_+^*(t) = 2\mathrm{Re}\left(c_+(t)\right)$. While $c_-(t)$ and $c_+(t)$ are complex-valued functions of time, because they are complex conjugates, their sum is a real-valued function of time.

(c) Signal $x_c(t)$ (fundamental frequency $f_0 = 2$ Hz) can be written as:

$$x_c(t) = \underbrace{2e^{j(\pi)}}_{\vec{X}_0}\, e^{j0t} + \underbrace{(2e^{j(-\frac{\pi}{2})} + \tfrac{3}{2}e^{j(\frac{\pi}{4})})}_{\frac{1}{2}\vec{X}_1}\, e^{j4\pi t} + \underbrace{(2e^{j(\frac{\pi}{2})} + \tfrac{3}{2}e^{j(-\frac{\pi}{4})})}_{\frac{1}{2}\vec{X}_{-1}}\, e^{-j4\pi t}$$

$$
\begin{aligned}
\vec{X}_0 &= -2\\
\tfrac{1}{2}\vec{X}_1 &= -2j + \tfrac{3}{2}\left(\tfrac{1}{2}\sqrt{2} + j\tfrac{1}{2}\sqrt{2}\right)\\
&= \tfrac{3}{4}\sqrt{2} + j\left(\tfrac{3}{4}\sqrt{2} - 2\right)\\
\tfrac{1}{2}\vec{X}_{-1} = \tfrac{1}{2}\vec{X}_1^* &= 2j + \tfrac{3}{2}\left(\tfrac{1}{2}\sqrt{2} - j\tfrac{1}{2}\sqrt{2}\right)\\
&= \tfrac{3}{4}\sqrt{2} - j\left(\tfrac{3}{4}\sqrt{2} - 2\right)
\end{aligned}
$$

All complex conjugate phasor pairs are drawn on the right.



Note that using the complex conjugate phasor pairs $\{\frac{1}{2}\vec{X}_k, \frac{1}{2}\vec{X}_{-k}\}$ the double-sided amplitude and phase spectra of periodic signal $x(t)$ can be directly obtained. Plotting, respectively, half of the phasor magnitude $\frac{1}{2}|\vec{X}_k|$ and phase angle $\angle\vec{X}_k$ as a function of kf_0, with $k \in \mathbb{Z}$ and f_0 the signal's fundamental frequency in [Hz], yields the double-sided amplitude and phase spectra.

Note that periodic signal $x(t)$ can be written as the sum of (here K) complex conjugate phasor pairs $\{\frac{1}{2}\vec{X}_k, \frac{1}{2}\vec{X}_{-k}\}$ multiplied by the corresponding complex conjugate exponential basis function pair with frequency kf_0:

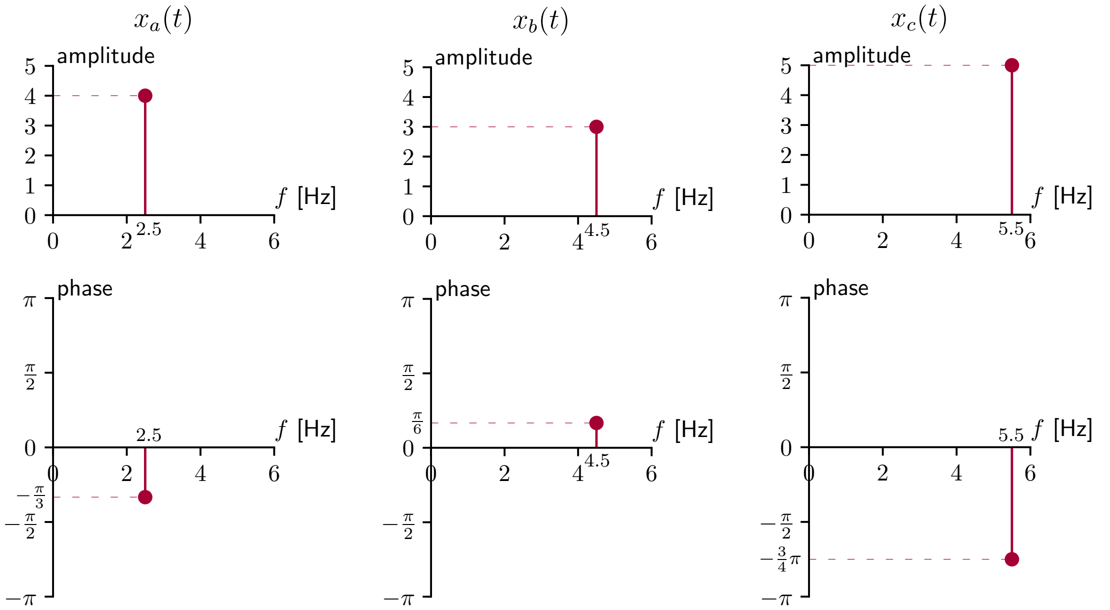
$$x(t) = \sum_{k=0}^{K-1}\left(\tfrac{1}{2}\vec{X}_k e^{j2\pi k f_0 t} + \tfrac{1}{2}\vec{X}_{-k}e^{-j2\pi k f_0 t}\right)$$

s.2-16

(a)
$$
\begin{aligned}
x_a(t) &= 4\cos\left(5\pi t - \tfrac{\pi}{3}\right) = \mathrm{Re}\left(4e^{j(5\pi t - \frac{\pi}{3})}\right) = \mathrm{Re}\left(4e^{j(-\frac{\pi}{3})}e^{j5\pi t}\right)\\
x_b(t) &= 3\cos\left(9\pi t + \tfrac{\pi}{6}\right) = \mathrm{Re}\left(3e^{j(9\pi t + \frac{\pi}{6})}\right) = \mathrm{Re}\left(3e^{j(\frac{\pi}{6})}e^{j9\pi t}\right)\\
x_c(t) &= 5\sin\left(11\pi t - \tfrac{\pi}{4}\right) = 5\cos\left(11\pi t - \tfrac{3\pi}{4}\right) = \mathrm{Re}\left(5e^{j(11\pi t - \frac{3\pi}{4})}\right) = \mathrm{Re}\left(5e^{j(-\frac{3\pi}{4})}e^{j11\pi t}\right)\\
x_d(t) &= 4\cos\left(5\pi t - \tfrac{\pi}{3}\right) + 3\cos\left(9\pi t + \tfrac{\pi}{6}\right) = \mathrm{Re}\left(4e^{j(5\pi t - \frac{\pi}{3})} + 3e^{j(9\pi t + \frac{\pi}{6})}\right)\\
&= \mathrm{Re}\left(4e^{j(-\frac{\pi}{3})}e^{j5\pi t} + 3e^{j(\frac{\pi}{6})}e^{j9\pi t}\right)\\
x_e(t) &= 4\cos\left(5\pi t - \tfrac{\pi}{3}\right) + 5\cos\left(11\pi t - \tfrac{3\pi}{4}\right) = \mathrm{Re}\left(4e^{j(5\pi t - \frac{\pi}{3})} + 5e^{j(11\pi t - \frac{3\pi}{4})}\right)\\
&= \mathrm{Re}\left(4e^{j(-\frac{\pi}{3})}e^{j5\pi t} + 5e^{j(-\frac{3\pi}{4})}e^{j11\pi t}\right)\\
x_f(t) &= 3\cos\left(9\pi t + \tfrac{\pi}{6}\right) + 5\cos\left(11\pi t - \tfrac{3\pi}{4}\right) = \mathrm{Re}\left(3e^{j(9\pi t + \frac{\pi}{6})} + 5e^{j(11\pi t - \frac{3\pi}{4})}\right)\\
&= \mathrm{Re}\left(3e^{j(\frac{\pi}{6})}e^{j9\pi t} + 5e^{j(-\frac{3\pi}{4})}e^{j11\pi t}\right)
\end{aligned}
$$

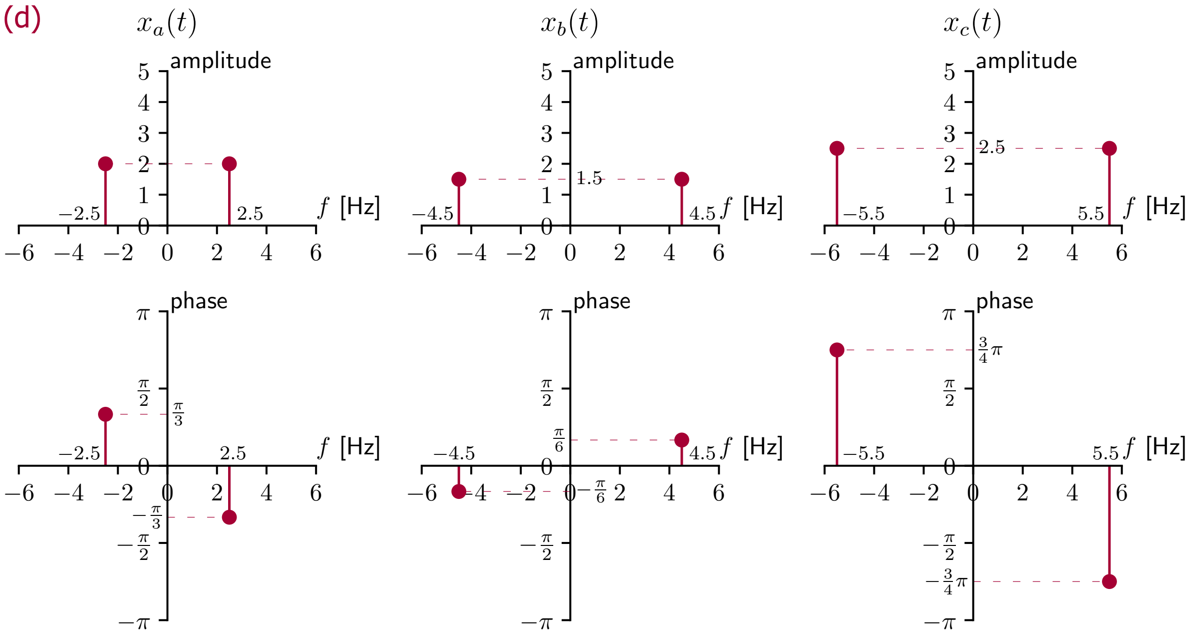
(b)
$$
\begin{aligned}
x_a(t) &= 2e^{j(5\pi t - \frac{\pi}{3})} + 2e^{-j(5\pi t - \frac{\pi}{3})} = 2e^{j(-\frac{\pi}{3})}e^{j5\pi t} + 2e^{j(\frac{\pi}{3})}e^{-j5\pi t}\\
x_b(t) &= 1.5e^{j(9\pi t + \frac{\pi}{6})} + 1.5e^{-j(9\pi t + \frac{\pi}{6})} = 1.5e^{j(\frac{\pi}{6})}e^{j9\pi t} + 1.5e^{j(-\frac{\pi}{6})}e^{-j9\pi t}\\
x_c(t) &= 2.5e^{j(11\pi t - \frac{3\pi}{4})} + 2.5e^{-j(11\pi t - \frac{3\pi}{4})} = 2.5e^{j(-\frac{3\pi}{4})}e^{j11\pi t} + 2.5e^{j(\frac{3\pi}{4})}e^{-j11\pi t}\\
x_d(t) &= 2e^{j(5\pi t - \frac{\pi}{3})} + 2e^{-j(5\pi t - \frac{\pi}{3})} + 1.5e^{j(9\pi t + \frac{\pi}{6})} + 1.5e^{-j(9\pi t + \frac{\pi}{6})}\\
&= 2e^{j(-\frac{\pi}{3})}e^{j5\pi t} + 2e^{j(\frac{\pi}{3})}e^{-j5\pi t} + 1.5e^{j(\frac{\pi}{6})}e^{j9\pi t} + 1.5e^{j(-\frac{\pi}{6})}e^{-j9\pi t}\\
x_e(t) &= 2e^{j(5\pi t - \frac{\pi}{3})} + 2e^{-j(5\pi t - \frac{\pi}{3})} + 2.5e^{j(11\pi t - \frac{3\pi}{4})} + 2.5e^{-j(11\pi t - \frac{3\pi}{4})}\\
&= 2e^{j(-\frac{\pi}{3})}e^{j5\pi t} + 2e^{j(\frac{\pi}{3})}e^{-j5\pi t} + 2.5e^{j(-\frac{3\pi}{4})}e^{j11\pi t} + 2.5e^{j(\frac{3\pi}{4})}e^{-j11\pi t}\\
x_f(t) &= 1.5e^{j(9\pi t + \frac{\pi}{6})} + 1.5e^{-j(9\pi t + \frac{\pi}{6})} + 2.5e^{j(11\pi t - \frac{3\pi}{4})} + 2.5e^{-j(11\pi t - \frac{3\pi}{4})}\\
&= 1.5e^{j(\frac{\pi}{6})}e^{j9\pi t} + 1.5e^{j(-\frac{\pi}{6})}e^{-j9\pi t} + 2.5e^{j(-\frac{3\pi}{4})}e^{j11\pi t} + 2.5e^{j(\frac{3\pi}{4})}e^{-j11\pi t}
\end{aligned}
$$

(c) Note that in order to sketch the amplitude and phase spectra of $x_c(t)$, the signal was first written as a cosine, as in (a).



The single-sided amplitude and phase spectra of signals $x_d(t)$, $x_e(t)$ and $x_f(t)$ can be directly constructed by combining the amplitude and phase spectra of their two individual signal components into a single amplitude plot and a single phase plot, respectively. Therefore, these spectra are not shown separately.

(d)



The double-sided amplitude and phase spectra of signals $x_d(t)$, $x_e(t)$ and $x_f(t)$ can be directly constructed by combining the amplitude and phase spectra of their two individual signal components into a single amplitude plot and a single phase plot, respectively. Therefore, these spectra are not shown separately.

S.2-17

The correct *single*-sided spectrum is spectrum (II).

Using $\cos^2 u = \frac{1}{2}\big(1+\cos(2u)\big)$, (A.4), $\sin u = \cos\big(u - \frac{\pi}{2}\big)$ and $\cos(u \pm \pi) = -\cos u$:

$$\begin{aligned} x(t) &= 2\cos^2(2\pi t) + 4\cos\big(8\pi t + \tfrac{\pi}{2}\big) + \sin\big(4\pi t - \tfrac{\pi}{2}\big) \\ &= 2\cdot\tfrac{1}{2}\big(1+\cos(4\pi t)\big) + 4\cos\big(8\pi t + \tfrac{\pi}{2}\big) + \cos\big(4\pi t - \tfrac{\pi}{2} - \tfrac{\pi}{2}\big) \\ &= 1 + \cos(4\pi t) + 4\cos\big(8\pi t + \tfrac{\pi}{2}\big) - \cos(4\pi t) \\ &= 1\cos(0\pi t + 0) + 4\cos\big(8\pi t + \tfrac{\pi}{2}\big) \end{aligned}$$

Signal $x(t)$ has an average of 1 and one sinusoidal component with frequency $f_1 = 4$ Hz.

$$x(t) = \mathrm{Re}\big(1e^{j(0)}e^{j0t} + 4e^{j\frac{\pi}{2}}e^{j8\pi t}\big)$$

S.2-18

(a) Using $\sin u = \cos\big(u - \frac{\pi}{2}\big)$:

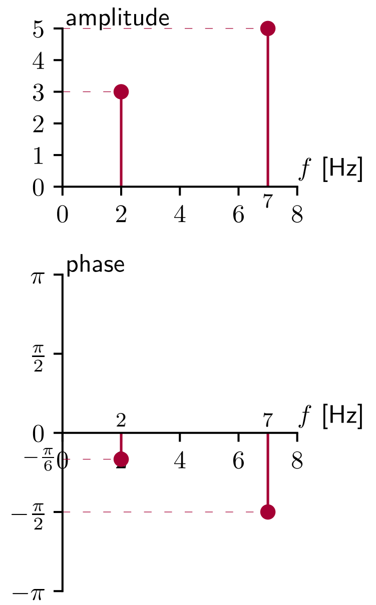
$$\begin{aligned} x(t) &= 3\cos\big(4\pi t - \tfrac{\pi}{6}\big) + 5\sin(14\pi t) \\ &= 3\cos\big(4\pi t - \tfrac{\pi}{6}\big) + 5\cos\big(14\pi t - \tfrac{\pi}{2}\big) \\ &= 3\cos\big(2(1)(2\pi)t - \tfrac{\pi}{6}\big) + 5\cos\big(7(1)(2\pi)t - \tfrac{\pi}{2}\big) \end{aligned}$$

Hence: signal $x(t)$ is periodic with $f_0 = 1$ Hz, $T_0 = 1$ s

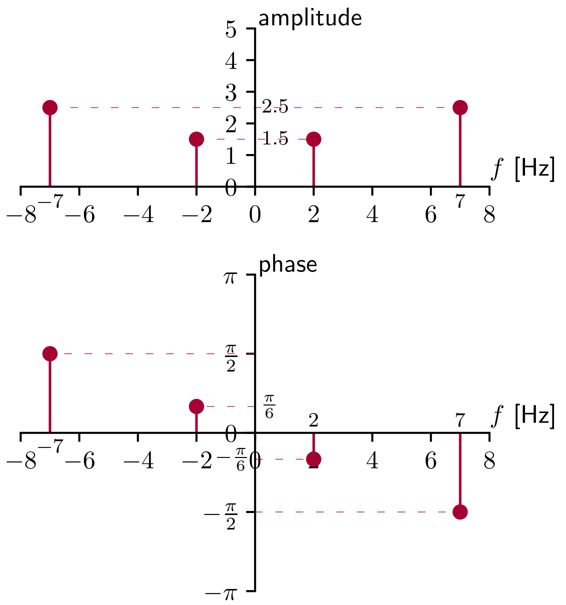
(b)
$$\begin{aligned} x(t) &= \mathrm{Re}\big(3e^{j(4\pi t - \frac{\pi}{6})} + 5e^{j(14\pi t - \frac{\pi}{2})}\big) \\ &= \mathrm{Re}\big(3e^{j(-\frac{\pi}{6})}e^{j4\pi t} + 5e^{j(-\frac{\pi}{2})}e^{j14\pi t}\big) \end{aligned}$$

(d)
$$\begin{aligned} x(t) &= 1.5e^{j(4\pi t - \frac{\pi}{6})} + 1.5e^{-j(4\pi t - \frac{\pi}{6})} \\ &\quad + 2.5e^{j(14\pi t - \frac{\pi}{2})} + 2.5e^{-j(14\pi t - \frac{\pi}{2})} \\ &= 1.5e^{j(-\frac{\pi}{6})}e^{j4\pi t} + 1.5e^{j(\frac{\pi}{6})}e^{-j4\pi t} \\ &\quad + 2.5e^{j(-\frac{\pi}{2})}e^{j14\pi t} + 2.5e^{j(\frac{\pi}{2})}e^{-j14\pi t} \end{aligned}$$

(c)



(e)



S.2-19

The correct *double*-sided spectrum is spectrum (III).

Using $\sin u = \cos\big(u - \frac{\pi}{2}\big)$ and $\cos u \cos v = \frac{1}{2}\big(\cos(u-v) + \cos(u+v)\big)$, (A.9):

$$x(t) = 8\sin\left(6\pi t + \tfrac{\pi}{2}\right)\cos(10\pi t) = 8\cos\left(6\pi t + \tfrac{\pi}{2} - \tfrac{\pi}{2}\right)\cos(10\pi t)$$
$$= 8\cos(6\pi t)\cos(10\pi t) = 8\cdot\tfrac{1}{2}\big(\cos(-4\pi t) + \cos(16\pi t)\big) = 4\cos(4\pi t) + 4\cos(16\pi t)$$

Signal $x(t)$ has two sinusoidal components with frequencies $f_1 = 2$ Hz and $f_2 = 8$ Hz.

$$x(t) = 2e^{j4\pi t} + 2e^{-j4\pi t} + 2e^{j16\pi t} + 2e^{-j16\pi t}$$

S.2-20

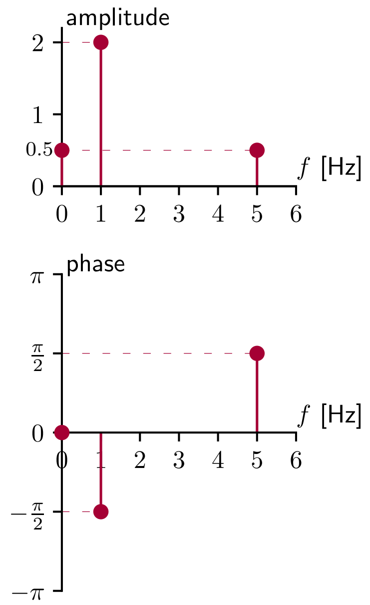
Using $\sin^2 u = \frac{1}{2}(1 - \cos(2u))$, (A.5), and $\cos(u \pm \pi) = -\cos u$:

$$x(t) = \sin^2\left(5\pi t - \tfrac{\pi}{4}\right) + 2\cos\left(2\pi t - \tfrac{\pi}{2}\right) = \tfrac{1}{2}\left(1 - \cos\left(10\pi t - \tfrac{\pi}{2}\right)\right) + 2\cos\left(2\pi t - \tfrac{\pi}{2}\right)$$
$$= \tfrac{1}{2} + \tfrac{1}{2}\cos\left(10\pi t - \tfrac{3\pi}{2}\right) + 2\cos\left(2\pi t - \tfrac{\pi}{2}\right) = \tfrac{1}{2} + 2\cos\left(2\pi t - \tfrac{\pi}{2}\right) + \tfrac{1}{2}\cos\left(10\pi t + \tfrac{\pi}{2}\right)$$

where in the last step 2π was added to the phase of the cosine with frequency $f_1 = 5$ Hz. Remember that the phase must always lie within $[-\pi, \pi]$ when drawing the phase spectrum.

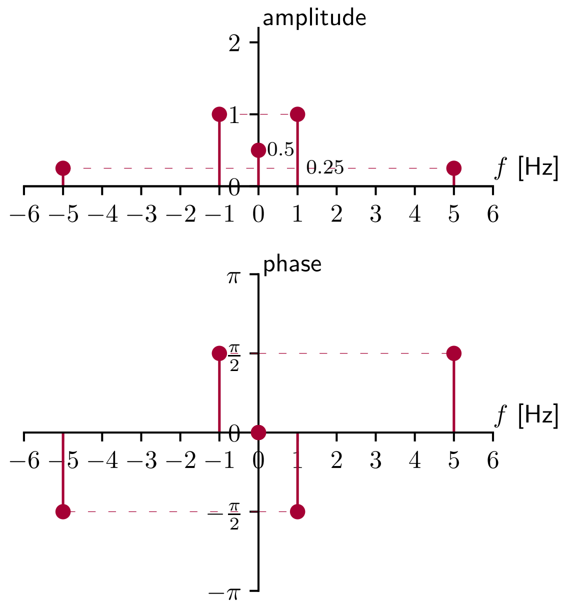
(a) $$x(t) = \mathrm{Re}\Big(\tfrac{1}{2}e^{j(0)} + 2e^{j(2\pi t - \frac{\pi}{2})} + \tfrac{1}{2}e^{j(10\pi t + \frac{\pi}{2})}\Big) = \mathrm{Re}\Big(\tfrac{1}{2}e^{j(0)} + 2e^{j(-\frac{\pi}{2})}e^{j2\pi t} + \tfrac{1}{2}e^{j\frac{\pi}{2}}e^{j10\pi t}\Big)$$

(b)



(c) $$x(t) = \tfrac{1}{2}e^{j(0)} + e^{j(2\pi t - \frac{\pi}{2})} + e^{-j(2\pi t - \frac{\pi}{2})} + \tfrac{1}{4}e^{j(10\pi t + \frac{\pi}{2})} + \tfrac{1}{4}e^{-j(10\pi t + \frac{\pi}{2})}$$
$$= \tfrac{1}{2}e^{j(0)} + e^{j(-\frac{\pi}{2})}e^{j2\pi t} + e^{j\frac{\pi}{2}}e^{-j2\pi t} + \tfrac{1}{4}e^{j\frac{\pi}{2}}e^{j10\pi t} + \tfrac{1}{4}e^{j(-\frac{\pi}{2})}e^{-j10\pi t}$$

(d)



S.2-21

The correct *double*-sided spectrum is spectrum (II).

Using $\cos^2 u = \frac{1}{2}(1 + \cos(2u))$, (A.4), and $\sin u = \cos\left(u - \frac{\pi}{2}\right)$:

$$x(t) = -2 + 4\cos^2(4\pi t) + 8\sin\left(6\pi t + \tfrac{\pi}{4}\right)$$
$$= -2 + 4\cdot\tfrac{1}{2}(1 + \cos(8\pi t)) + 8\cos\left(6\pi t + \tfrac{\pi}{4} - \tfrac{\pi}{2}\right)$$
$$= -2 + 2 + 2\cos(8\pi t) + 8\cos\left(6\pi t - \tfrac{\pi}{4}\right) = 8\cos\left(6\pi t - \tfrac{\pi}{4}\right) + 2\cos(8\pi t)$$

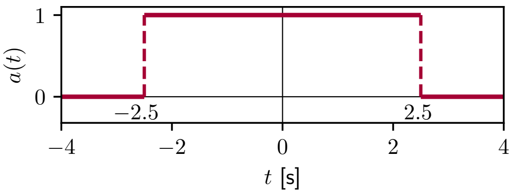
Signal $x(t)$ has two sinusoidal components with frequencies $f_1 = 3$ Hz and $f_2 = 4$ Hz.

$$x(t) = 4e^{j(-\frac{\pi}{4})}e^{j6\pi t} + 4e^{j\frac{\pi}{4}}e^{-j6\pi t} + e^{j8\pi t} + e^{-j8\pi t}$$

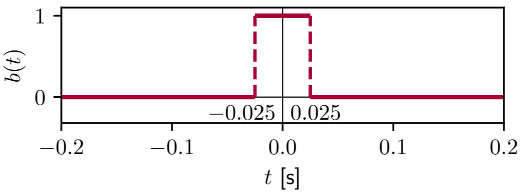
s | 2.4 Signal operations

s.2-22

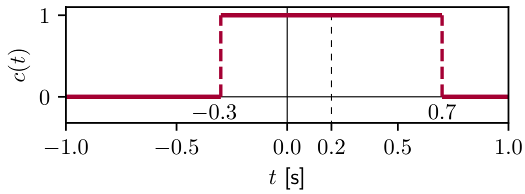
(a) $\Pi(0.2t) = \Pi\left(\frac{t}{5}\right)$, i.e., a unit pulse with a width of 5 s centered at $t = 0$ s.



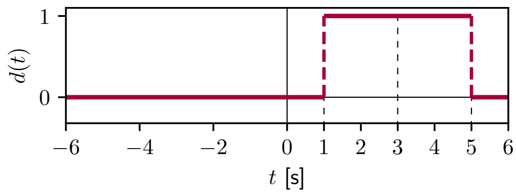
(b) $\Pi(20t) = \Pi\left(\frac{t}{0.05}\right)$, i.e., a unit pulse with a width of 0.05 s centered at $t = 0$ s.



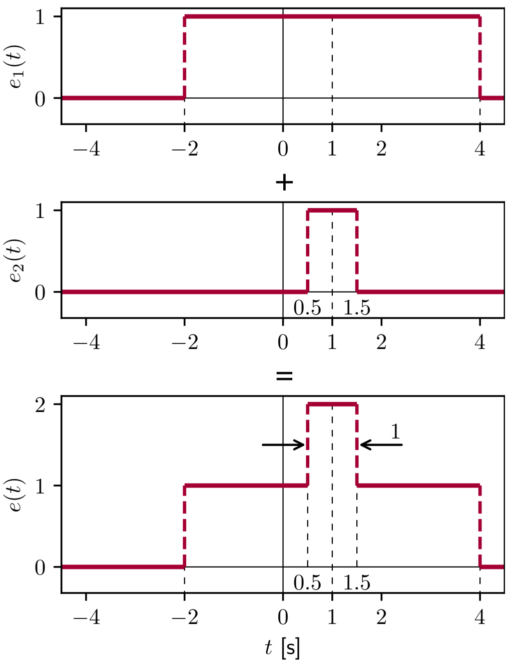
(c) A unit pulse with a width of 1 s centered at $t = \frac{1}{5}$ s $= 0.2$ s.



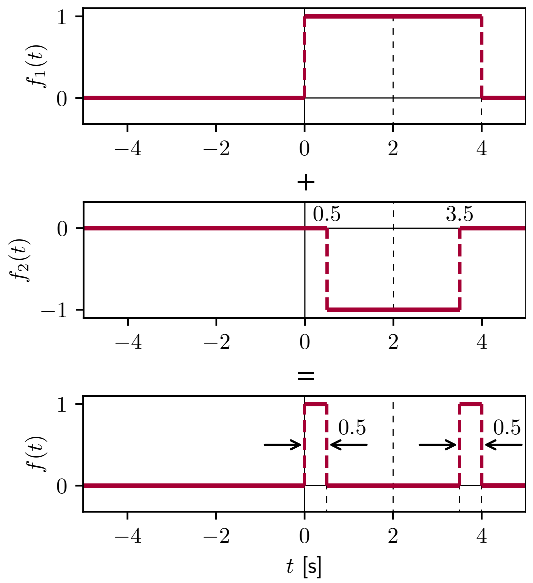
(d) A unit pulse with a width of 4 s centered at $t = 3$ s.



(e) The sum of a unit pulse with a width of 6 s centered at $t = 1$ s and a unit pulse with a width of 1 s centered at $t = 1$ s.



(f) $\Pi\left(\frac{t-2}{4}\right) - \Pi\left(\frac{t-2}{3}\right) =$
$\Pi\left(\frac{t-2}{4}\right) + \left(-\Pi\left(\frac{t-2}{3}\right)\right)$, i.e., a unit pulse with a width of 3 s centered at $t = 2$ s subtracted from a unit pulse with a width of 4 s centered at $t = 2$ s.

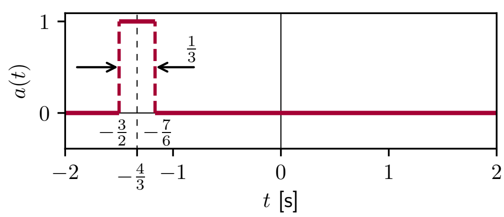


s.2-23

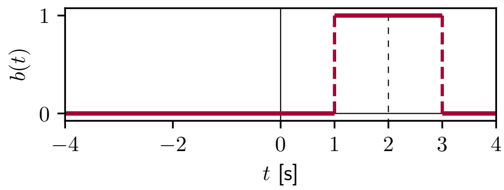
Let us first rewrite the unit pulse function: $\Pi(\rho - \sigma t) = \Pi\left(-\sigma\left(t - \frac{\rho}{\sigma}\right)\right)$
The illustration shows that the pulse function is centered at $t = 0$ s, hence $\rho = 0$. As the pulse function is an even function, i.e., mirroring the function about $t = 0$ s has no effect, the sign of σ cannot be determined. Hence:
For ρ: statement (3) The parameter is equal to zero (parameter $= 0$), is correct.
For σ: statement (4) The parameter's sign cannot be determined, is correct.

S.2-24

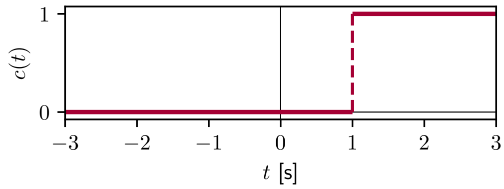
(a) $\Pi(3t+4) = \Pi\left(3\left(t+\frac{4}{3}\right)\right) = \Pi\left(\frac{t+4/3}{1/3}\right)$, i.e., a unit pulse with a width of $\frac{1}{3}$ s centered at $t = -\frac{4}{3}$ s.



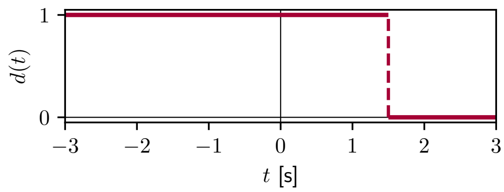
(b) $\Pi(-0.5t+1) = \Pi\left(-\frac{1}{2}(t-2)\right) = \Pi\left(-\frac{t-2}{2}\right)$, which equals $\Pi\left(\frac{t-2}{2}\right)$, because the unit pulse function is an even function, i.e., $x(-t) = x(t)$ (mirroring about $t = 0$ s has no effect).
$\Pi\left(\frac{t-2}{2}\right)$ is a unit pulse with a width of 2 s centered at $t = 2$ s.



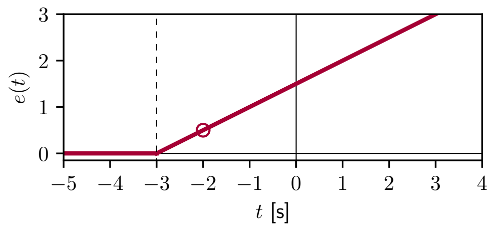
(c) $u\left(\frac{t-1}{2}\right)$, i.e., the step function is shifted to the *right* by 1 s. Note that the 2 in the denominator has no effect.



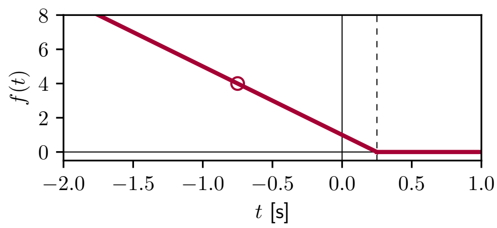
(d) $u(-2t+3) = u\left(-2(t-1.5)\right)$, i.e., the step function is first mirrored about $t = 0$ s, and then shifted to the *right* by 1.5 s. As a result of the 'flip', the function is 1 for $t < 1.5$ s and 0 for $t > 1.5$ s. Note that the factor 2 has no effect.



(e) $r\left(\frac{t+3}{2}\right) = r\left(\frac{1}{2}(t+3)\right)$. The factor $\frac{1}{2}$ indicates that the signal runs twice as slowly, meaning that the function rises less rapidly in time. $t_0 = +3$ s, hence the signal is shifted to the *left* by 3 s.
Note that the circle has been added to check that the signal rises by 0.5 units every second.



(f) $r(-4t+1) = r\left(-4\left(t-\frac{1}{4}\right)\right)$. The ramp function is first mirrored about $t = 0$ s, as $-4 < 0$. The (absolute) value of 4 indicates that the signal runs four times as fast, meaning that the function rises more rapidly in time. $t_0 = -\frac{1}{4}$ s, hence the signal is shifted to the *right* by $\frac{1}{4}$ s.
Note that the circle has been added to check that the signal rises by 4 units every second.



S.2-25

The operated unit pulse function, let us call it $p(t)$, has amplitude $A = 1$ and a width of 2 s, and is centered at $t = 3$ s. Hence: $p(t) = \Pi\left(\frac{t-3}{2}\right)$
The operated unit ramp function, let us call it $m(t)$, is mirrored about $t = 0$ s, rises by 1 unit every second, and is shifted to the *left* by 1 s, i.e., $t_0 = +1$ s. Hence: $m(t) = r\left(-1(t+1)\right)$
Combining $m(t)$ and $p(t)$ gives us the expression for $x(t)$:

$$x(t) = m(t) + p(t) = r(-t-1) + \Pi\left(\frac{t-3}{2}\right)$$

S.2-26

(a)-(c) The solutions to (a)-(c) are given in the following table:

	T_0 in [s]	f_0 in [Hz]	unknown in [rad]	θ in [rad]	t_0 in [s]
(a)	1	1	$a = \frac{\pi}{2}$	$\frac{\pi}{2}$	$\frac{1}{4}$
(b)	$\frac{2}{3}$	$\frac{3}{2}$	$b = \frac{\pi}{2}$	$\frac{\pi}{2}$	$\frac{1}{6}$
(c)	$\frac{2}{5}$	$\frac{5}{2}$	$c = \frac{\pi}{2}$	$\frac{\pi}{2}$	$\frac{1}{10}$

We see that, for a function that consists of a single sinusoid, adding $\frac{\pi}{2}$ rad to the phase of the signal (i.e., a phase shift of $\frac{\pi}{2}$ rad) shifts that signal a quarter wavelength in time to the left, no matter what its frequency (or amplitude). This is because $t_0 = \frac{\theta}{2\pi} T_0$ s, see page 15 of the companion book, Theory, so with $t_0 = \frac{T_0}{4}$ s, $\theta = 2\pi \frac{t_0}{T_0} = 2\pi \frac{1}{4} = \frac{\pi}{2}$ rad.

(d) Signal $x_d(t)$ consists of two sinusoidal components with frequencies, respectively, $f_1 = 3$ Hz and $f_2 = 7.5$ Hz. The fundamental frequency of $x_d(t)$ is then $f_0 = \frac{3}{2}$ Hz, the fundamental period $T_0 = \frac{2}{3}$ s. The two unknowns d_1 and d_2 can be obtained as follows:

$$\begin{aligned} y_d(t) = x_d\left(t + \tfrac{T_0}{4}\right) &= 2\cos\left(6\pi\left(t + \tfrac{1}{6}\right) + \tfrac{\pi}{4}\right) - 3\sin\left(15\pi\left(t + \tfrac{1}{6}\right)\right) \\ &= 2\cos\left(6\pi t + \pi + \tfrac{\pi}{4}\right) - 3\sin\left(15\pi t + \tfrac{5\pi}{2}\right) \end{aligned}$$

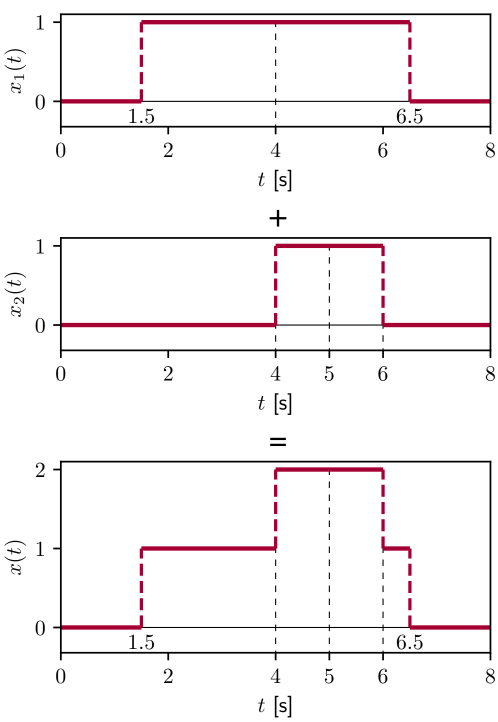
We obtain $d_1 = \pi$ and $d_2 = \frac{5\pi}{2}$. For $x_d(t)$, the phase shifts of its components are *not* equal to $\frac{\pi}{2}$ rad (as in (a)-(c)). This is because the frequencies f_1 and f_2 of the components are not equal to the fundamental frequency f_0 of their sum, $x_d(t)$. In fact, because $f_1 = 2f_0$ we get for the first component a phase shift θ of $2\frac{\pi}{2} = \pi$; and since $f_2 = 5f_0$ we get for the second component a phase shift θ of $5\frac{\pi}{2} = \frac{5\pi}{2}$. The time shift t_0 equals $\frac{1}{6}$ s.

S.2-27

(a) $x(t) = \Pi\left(\frac{t-4}{5}\right) + \Pi\left(\frac{t-5}{2}\right)$ is the sum of a unit pulse with a width of 5 s centered at $t = 4$ s and a unit pulse with a width of 2 s centered at $t = 5$ s.

(b) Looking at the sketch:
$x(t) = u(t-1.5) + u(t-4) - u(t-6) - u(t-6.5)$

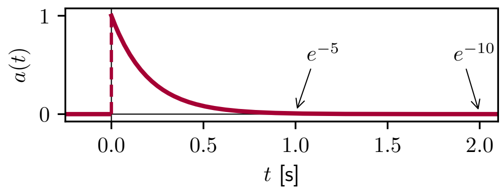
(c)
$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\big(x(t)\big) &= \frac{\mathrm{d}}{\mathrm{d}t}\big(u(t-1.5) + u(t-4) \\ &\qquad - u(t-6) - u(t-6.5)\big) \\ &= \delta(t-1.5) + \delta(t-4) \\ &\qquad - \delta(t-6) - \delta(t-6.5) \end{aligned}$$



s|2.5 Energy and power

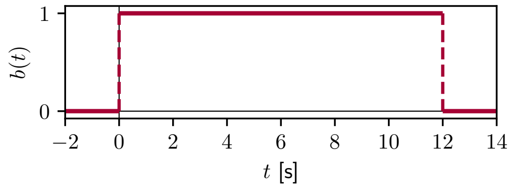
S.2-28

(a) Note that this signal has been treated in general form in Example 2.9. Here, $A = 1$ and $\alpha = 5$.



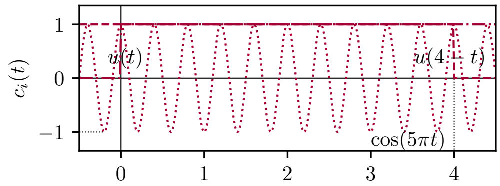
$$E_a = \lim_{T\to\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}} x^2(t)\,\mathrm{d}t = \lim_{T\to\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}} \left(e^{-5t}u(t)\right)^2 \mathrm{d}t$$

$$= \lim_{T\to\infty}\int_{0}^{\frac{T}{2}} e^{-10t}\,\mathrm{d}t = \frac{1}{-10}\lim_{T\to\infty}(e^{-5T}-1) = \frac{1}{10}$$



(b) $E_b = \lim_{T\to\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}} x^2(t)\,\mathrm{d}t = \int_0^{12}(1)^2\,\mathrm{d}t = 12$

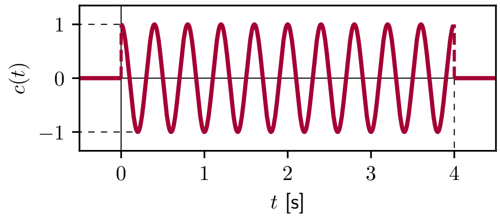
(or simply, A^2 (amplitude squared) multiplied by the signal width)



(c) $E_c = \lim_{T\to\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}} x^2(t)\,\mathrm{d}t = \int_0^4 \cos^2(5\pi t)\,\mathrm{d}t$

$$= \frac{1}{2}\int_0^4 \left(1+\cos(10\pi t)\right)\mathrm{d}t$$

$$= \frac{1}{2}\left[t + \frac{1}{10\pi}\sin(10\pi t)\right]_0^4 = \frac{1}{2}\cdot 4 = 2$$

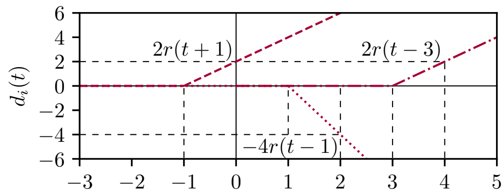


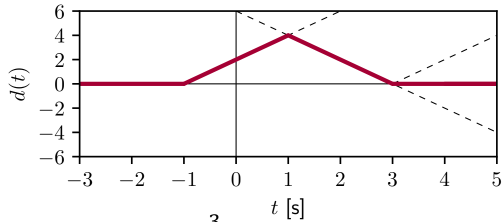
(d) First put the components in their order of appearance:

$2r(t-3) + 2r(t+1) - 4r(t-1)$

$\quad = 2r(t+1) - 4r(t-1) + 2r(t-3)$

The first ramp function is zero before $t = -1$ s and then rises by 2 units every second. The second ramp function is zero before $t = 1$ s and then descends by 4 units every second. The third ramp function is zero before $t = 3$ s and then rises by 2 units every second.





$$E_d = \lim_{T\to\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}} x^2(t)\,\mathrm{d}t$$

$$= \int_{-1}^{1}(2+2t)^2\,\mathrm{d}t + \int_1^3 (6-2t)^2\,\mathrm{d}t = \int_{-1}^{1}(4+8t+4t^2)\,\mathrm{d}t + \int_1^3(36-24t+4t^2)\,\mathrm{d}t$$

$$= \left[4t+4t^2+\tfrac{4}{3}t^3\right]_{-1}^{1} + \left[36t-12t^2+\tfrac{4}{3}t^3\right]_1^3 = \frac{64}{3}$$

S.2-29

See Example 2.10 for the average power of a sinusoidal signal.

(a)-(c) All three signals have amplitude $A = 1$. Their average power P equals $\frac{A^2}{2} = \frac{1^2}{2} = \frac{1}{2}$.

(d) $x_d(t) = x_a(t) + x_b(t)$, then is the power of $x_d(t)$ equal to the power of $x_a(t)$ plus the power of $x_b(t)$?

$$P_{x_d} = \lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}(x_a^2(t)+x_b^2(t)+2x_a(t)x_b(t))\,\mathrm{d}t = \frac{1}{2}+\frac{1}{2}+\lim_{T\to\infty}\frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x_a(t)x_b(t)\,\mathrm{d}t$$

The latter integral contains a sine function and a cosine function that have a common base frequency (see Solution S.2-5(d)), so using the orthogonality properties, more specifically (A.13), we can safely say that the integral on the right-hand side equals zero. The sinusoids are *harmonically related*. Clearly then, we can add the powers of $x_a(t)$ and $x_b(t)$. Hence, $P_{x_d} = \frac{1}{2} + \frac{1}{2} = 1$.

(e) As shown in Solution S.2-5(e), signals $x_a(t)$ and $x_c(t)$ share a fundamental frequency f_0 of 2 Hz. Using orthogonality property (A.13): $P_{x_e} = \frac{1}{2} + \frac{1}{2} = 1$.

(f) As shown in Solution S.2-5(f), signals $x_b(t)$ and $x_c(t)$ share a fundamental frequency f_0 of 3 Hz. Using orthogonality property (A.12): $P_{x_f} = \frac{1}{2} + \frac{1}{2} = 1$.

S.2-30

See Example 2.10 for the average power of a sinusoidal signal.

(a) $P_{x_a} = \frac{A^2}{2} = \frac{4^2}{2} = \frac{16}{2} = 8$

(b) $P_{x_b} = \frac{A^2}{2} = \frac{3^2}{2} = \frac{9}{2} = 4.5$

(c) $P_{x_c} = \frac{A^2}{2} = \frac{5^2}{2} = \frac{25}{2} = 12.5$

(d) As shown in Solution S.2-6(d), signals $x_a(t)$ and $x_b(t)$ share a fundamental frequency f_0 of 0.5 Hz. Using orthogonality property (A.12): $P_{x_d} = 8 + 4.5 = 12.5$.

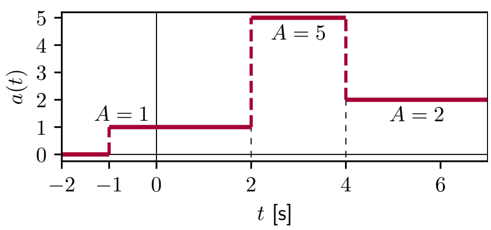
(e) As shown in Solution S.2-6(e), signals $x_a(t)$ and $x_c(t)$ share a fundamental frequency f_0 of 0.5 Hz. Using orthogonality property (A.13): $P_{x_e} = 8 + 12.5 = 20.5$.

(f) As shown in Solution S.2-6(f), signals $x_b(t)$ and $x_c(t)$ share a fundamental frequency f_0 of 0.5 Hz. Using orthogonality property (A.13): $P_{x_f} = 4.5 + 12.5 = 17$.

S.2-31

In exercises like these, it is recommended to first make a sketch of the signal (if the signal is not too challenging to sketch). Usually, from the sketch it becomes automatically clear whether the signal is an energy signal, a power signal, or neither.

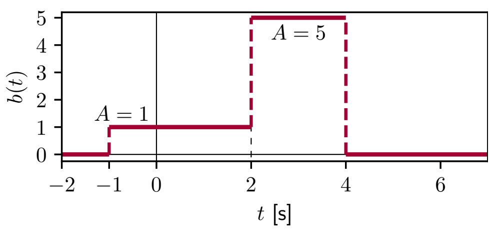
(a) The signal components are already in their order of appearance, so the signal can be sketched directly. As shown on the right, the signal attains a value of 2 from $t = 4$ s to $t = \infty$. Its energy is thus infinite; it is not an energy signal. The signal's average power can be computed, though, using:



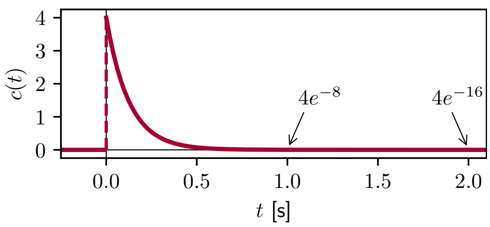
$$P_a = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x^2(t)\,\mathrm{d}t, \qquad \text{for a suitable time interval } T.$$

Note that, by approximation, the signal has a value of 0 for $t < 0$ s and a value of 2 for $t > 0$ s, since taking T to go to infinity makes the effects from $t = -1$ s to $t = 4$ s negligible (the signal is 2 for a much longer time than it is either 1 or 5). Hence, $P_a = \frac{0^2}{2} + \frac{2^2}{2} = 2$. The signal is thus a power signal.

(b) The sketch shows that the signal has finite energy, and is therefore an energy signal. Its energy can be computed by multiplying A^2 (amplitude squared) by the signal width:
$E_b = (1)^2 \cdot 3 + (5)^2 \cdot 2 = 53$



(c) Note that this signal has been treated in general form in Example 2.9. Here, $A = 4$ and $\alpha = 8$. Its shape is also comparable to the signal sketched in Solution S.2-28(a), but with different (still positive) values for A and α. Hence, it is an energy signal, as its energy is finite:



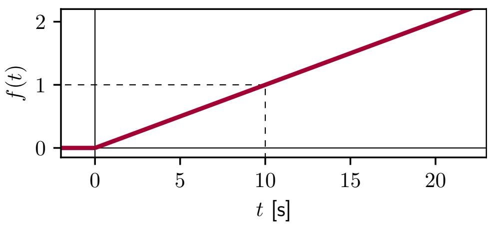
$$E_c = \lim_{T\to\infty} \int_{-\frac{T}{2}}^{\frac{T}{2}} \left(4e^{-8t}u(t)\right)^2 \mathrm{d}t = 4^2 \lim_{T\to\infty} \int_0^{\frac{T}{2}} e^{-16t}\,\mathrm{d}t = \frac{16}{-16} \lim_{T\to\infty} (e^{-8T} - 1) = 1$$

Or using the energy equation derived in Example 2.9: $E_c = \frac{A^2}{2\alpha} = \frac{4^2}{16} = 1$

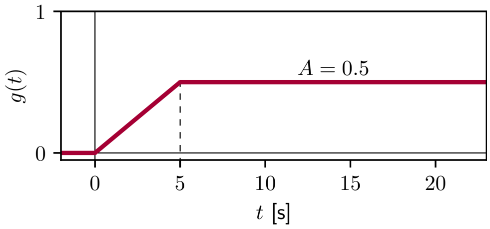
(d) Here, $d(t) = \left(4e^{-8t} + 3\right)u(t) = 4e^{-8t}u(t) + 3u(t)$. The first component equals the signal treated in (c) and has limited energy. The second component, however, equals 3 from $t = 0$ s to $t = \infty$; its energy is therefore unlimited. Hence, the signal is not an energy signal. Somewhat comparable to the signal in (a), this signal has a value of 0 for $t < 0$ s and approximately 3 for $t > 0$ s. Hence, it is a power signal with $P_d = \frac{0^2}{2} + \frac{3^2}{2} = 4.5$.

(e) Here, $e(t) = \left(3 - 4e^{-8t}\right)u(t) = 3u(t) - 4e^{-8t}u(t)$. The signal is 0 for $t < 0$ s and $x(t) \to 3$ when t becomes large. Hence, the signal has infinite energy. It is, however, a power signal with (again, just like in (d)) $P_e = \frac{0^2}{2} + \frac{3^2}{2} = 4.5$.

(f) The sketch shows that the ramp signal has infinite energy, as it goes on and on forever. It also has infinite average power, as it does not stop at some constant level but, as said, continues to grow forever. This ramp signal is an example of a signal that is neither energy, nor power.



(g) The sketch shows that the ramp signal is 'stopped' in its growth by the subtraction of the second ramp component, at $t = 5$ s, and it keeps a constant value of 0.5. Clearly, it is not an energy signal, but a power signal, with $P_g = \frac{0^2}{2} + \frac{0.5^2}{2} = 0.125$.



(h) Here, $h(t) = t^{-\frac{1}{6}}u(t-6) = \frac{1}{\sqrt[6]{t}}u(t-6)$. Even though it would be challenging to sketch $h(t)$, you do not really need a sketch to get a first idea of this signal. Just substitute some positive values of t (for $t \geq 6$) into the equation to see that it will become smaller for larger values of t. The rate at which it becomes smaller, however, is also decreasing rapidly. So, although clearly for everyone that $x(\infty) = 0$, the energy may still be very large. Let us compute it:

$$E_h = \lim_{T\to\infty} \int_{-\frac{T}{2}}^{\frac{T}{2}} x^2(t)\,\mathrm{d}t = \int_6^{\infty} \left(t^{-\frac{1}{6}}\right)^2 \mathrm{d}t = \int_6^{\infty} t^{-\frac{1}{3}}\,\mathrm{d}t = \left[\frac{3}{2}t^{\frac{2}{3}}\right]_6^{\infty} = \infty!$$

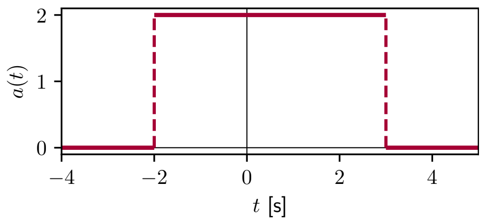
Hence, it is not an energy signal. However, it is not a power signal either, as its average power equals zero. This is an example of a signal that is neither energy, nor power.

S.2-32

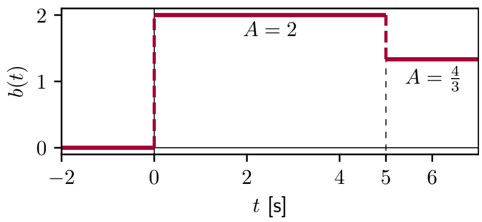
$x(t) = 3\cos\left(4\pi t - \frac{\pi}{6}\right) + 5\sin(14\pi t)$ is a periodic signal (see Solution S.2-18(a)). Hence, it is a power signal, and its energy is infinite. Its average power can be computed by taking the sum of the average powers of the individual components: $P_x = \frac{3^2}{2} + \frac{5^2}{2} = 17$.

S.2-33

(a) The sketch shows that the signal has finite energy, and is therefore an energy signal. Its energy can be computed by multiplying A^2 (amplitude squared) by the signal width:
$E_a = (2)^2 \cdot 5 = 20.$



(b) From the sketch it is clear that the signal has infinite energy. By approximation, the signal is 0 for any value of $t < 0$ s and equals $\frac{4}{3}$ for any value of $t > 0$ s. Hence, it is a power signal with
$P_b = \frac{0^2}{2} + \frac{\left(\frac{4}{3}\right)^2}{2} = \frac{8}{9}.$



(c) Here, sketching the signal is less straightforward. However, this signal is 0 for $t < 0$ s and also becomes 0 for $t \to \infty$, so for the time being let us assume it is an energy signal. Let us try to compute the energy by making use of the partial integration rule:
$\int_a^b \frac{\mathrm{d}f(x)}{\mathrm{d}x} g(x)\,\mathrm{d}x = [f(x)g(x)]_a^b - \int_a^b f(x)\frac{\mathrm{d}g(x)}{\mathrm{d}x}\,\mathrm{d}x.$

$$E_c = \lim_{T\to\infty} \int_{-\frac{T}{2}}^{\frac{T}{2}} x^2(t)\,\mathrm{d}t = \int_0^\infty (2te^{-3t})^2\,\mathrm{d}t = 4\int_0^\infty t^2 e^{-6t}\,\mathrm{d}t$$

Use $\frac{\mathrm{d}f(t)}{\mathrm{d}t} = e^{-6t}$, and $g(t) = t^2$, then it follows that $f(t) = -\frac{1}{6}e^{-6t}$ and $\frac{\mathrm{d}g(t)}{\mathrm{d}t} = 2t$:

$$E_c = 4\left[-\tfrac{1}{6}t^2 e^{-6t}\right]_0^\infty - 4\int_0^\infty \left(-\tfrac{1}{6}e^{-6t}\right) 2t\,\mathrm{d}t = 0 + \tfrac{4}{3}\int_0^\infty te^{-6t}\,\mathrm{d}t$$

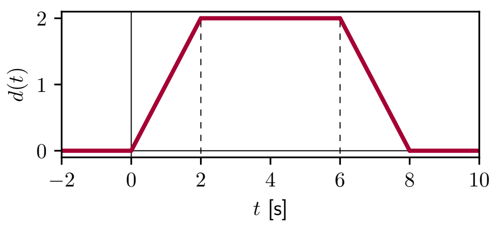
Use the partial integration rule again, with $f(t)$ and $\frac{\mathrm{d}f(t)}{\mathrm{d}t}$ the same as above, but now with $g(t) = t$ and $\frac{\mathrm{d}g(t)}{\mathrm{d}t} = 1$:

$$E_c = \tfrac{4}{3}\left[-\tfrac{1}{6}te^{-6t}\right]_0^\infty - \tfrac{4}{3}\int_0^\infty \left(-\tfrac{1}{6}e^{-6t}\right)\mathrm{d}t = 0 + \tfrac{2}{9}\int_0^\infty e^{-6t}\,\mathrm{d}t = \tfrac{2}{9}\left[-\tfrac{1}{6}e^{-6t}\right]_0^\infty = \tfrac{1}{27}$$

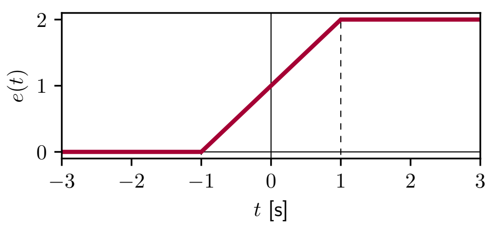
So, we get a finite value, and we are dealing (as expected) with an energy signal.

(d) The sketch shows that the signal has finite energy, and hence, is an energy signal:

$$E_d = \int_0^2 (t)^2\,\mathrm{d}t + \int_2^6 (2)^2\,\mathrm{d}t + \int_6^8 (8-t)^2\,\mathrm{d}t$$
$$= \left[\tfrac{1}{3}t^3\right]_0^2 + [4t]_2^6 + \left[64t - 8t^2 + \tfrac{1}{3}t^3\right]_6^8 = \tfrac{64}{3}$$

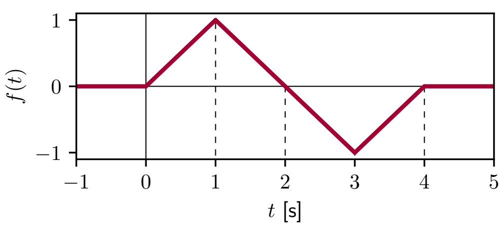


(e) From the sketch it is clear that the signal has infinite energy. By approximation, the signal is 0 for any value of $t < 0$ s and equals 2 for any value of $t > 0$ s. Hence, it is a power signal with
$P_e = \frac{0^2}{2} + \frac{2^2}{2} = 2.$

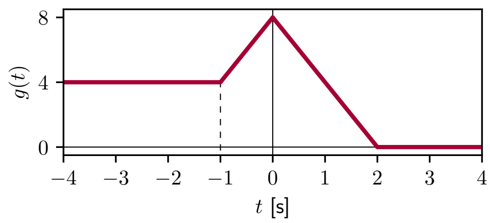


(f) The sketch shows that the signal has finite energy, and hence, is an energy signal:

$$E_f = \int_0^1 (t)^2\,\mathrm{d}t + \int_1^3 (2-t)^2\,\mathrm{d}t + \int_3^4 (t-4)^2\,\mathrm{d}t$$
$$= \left[\tfrac{1}{3}t^3\right]_0^1 + \left[4t - 2t^2 + \tfrac{1}{3}t^3\right]_1^3$$
$$+ \left[\tfrac{1}{3}t^3 - 4t^2 + 16t\right]_3^4 = \tfrac{4}{3}$$



(g) From the sketch it is clear that the signal has infinite energy. By approximation, the signal is 4 for any value of $t < 0$ s and equals 0 for any value of $t > 0$ s. Hence, it is a power signal with $P_g = \frac{4^2}{2} + \frac{0^2}{2} = 8$.

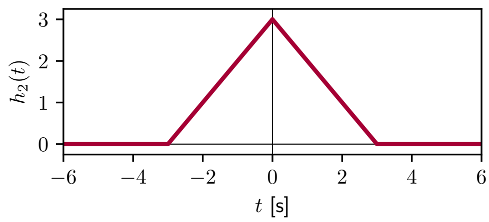


(h) Sketching this signal is somewhat less straightforward, but we can split up the signal into two components, the cosine function and the ramp functions:

$$h(t) = h_1(t) + h_2(t),$$

where $h_1(t) = 8\cos\left(6\pi t + \frac{\pi}{3}\right)$

and $h_2(t) = r(t+3) - 2r(t) + r(t-3)$.



From the sketch of $h_2(t)$ on the right, it is clear that $h_2(t)$ by itself would be an energy signal with finite energy and an average power of zero. However, $h_1(t)$ is a periodic signal, more specifically, a sinusoidal signal with an average power only depending on its amplitude. Hence, $h(t)$ is a power signal with $P_h = \frac{A_1^2}{2} = \frac{8^2}{2} = 32$.

Similar to the pulse in Example 2.11, $h_2(t)$ becomes irrelevant in this case.

S.2-34

(a) $a(t) = \cos(3\pi t) + \sin(4\pi t) = \cos(1.5(2\pi)t) + \sin\big(2(2\pi)t\big)$

$$= \cos(3(0.5)(2\pi)t) + \sin(4(0.5)(2\pi)t)$$

The two components share a fundamental frequency f_0 of 0.5 Hz. Hence, the signal is periodic with period $T_0 = 2$ s. Because the signal is periodic, it is a power signal. Its average power can be computed by taking the sum of the average powers of the individual components: $P_a = \frac{1^2}{2} + \frac{1^2}{2} = 1$.

(b) $b(t) = \sin(5t) + \cos(5\pi t) = \sin\left(\frac{5}{2\pi}(2\pi)t\right) + \cos\big(2.5(2\pi)t\big)$

Clearly, the signal components do not share a common frequency. The frequency of the sine component equals $\frac{5}{2\pi}$ Hz and the frequency of the cosine component equals 2.5 Hz. The ratio of the two frequencies is not a rational number, so the signal is *not* periodic.

Let us check whether it is an energy signal. By looking at the definition of $b(t)$, one can see that integrating $b^2(t)$ will yield infinity. In fact, both components (which are periodic) are power signals. Hence, $b(t)$ is not an energy signal.

Let us now try to compute the average power of $b(t)$:

$$P_b = \lim_{T\to\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} b^2(t)\,\mathrm{d}t = \lim_{T\to\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \big(\sin(5t) + \cos(5\pi t)\big)^2\,\mathrm{d}t$$

$$= \lim_{T\to\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \sin^2(5t)\,\mathrm{d}t + \lim_{T\to\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \cos^2(5\pi t)\,\mathrm{d}t + \lim_{T\to\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} 2\sin(5t)\cos(5\pi t)\,\mathrm{d}t$$

$$= \frac{1}{2} + \frac{1}{2} + \lim_{T\to\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \Big(\sin\big((5-5\pi)t\big) + \sin\big((5+5\pi)t\big)\Big)\,\mathrm{d}t$$

where we used (A.10). Note that the latter integral value is always smaller than or equal to 1, so when dividing by T and taking the limit $T \to \infty$, the result will be zero. Hence, the average power is again the summation of the two components.

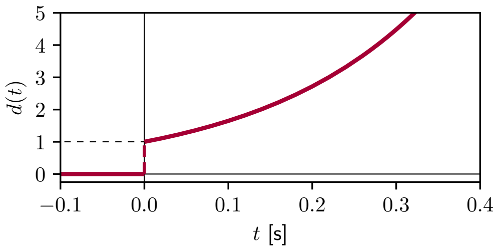
Thus, $b(t)$ is a power signal with $P_b = 1$.

(c) Note that this signal has been treated in general form in Example 2.9. Here, $A = 3$ and $\alpha = 2$. It is also comparable to the signal sketched in Solution S.2-28(a), but with different (still positive) values for A and α. As is clear from the comparable signal sketched in Solution S.2-28(a), this signal is *not* periodic.
Let us try to compute its energy:

$$E_c = \lim_{T\to\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}}(3e^{-2t}u(t))^2\,\mathrm{d}t = 3^2\lim_{T\to\infty}\int_0^{\frac{T}{2}} e^{-4t}\,\mathrm{d}t = \frac{9}{-4}\lim_{T\to\infty}(e^{-2T}-1) = \frac{9}{-4}(0-1) = \frac{9}{4}$$

Clearly, the energy is limited, and the signal is an energy signal.

(d) As can clearly be seen from the sketch on the right, the signal continues to grow for larger values of t. Hence, the signal is *not* periodic.
It is definitely not an energy signal, as its energy is infinite. It is also not a power signal, because the signal does not converge to some non-zero constant value, but continues to increase. This is a signal that is neither energy, nor power.



S.2-35

(a) $P = 1$ microWatt $= 10^{-6}$ W $= 0.000001$ W $\quad\rightarrow\quad P_{dB} = 10\log_{10}(10^{-6}) = -60$ dB W

(b) When the power of a signal, expressed in Watt, is doubled, $10\log_{10}(2) \approx 3$ dB in terms of dB W are added.

S.2-36

Statement A. is correct.

Signal $x(t) = 2\cos\left(100\pi t + \frac{\pi}{3}\right) + 4\sin(200\pi t) - 2\cos\left(400\pi t - \frac{\pi}{6}\right) - \cos(1000\pi t)$ is a summation of four sinusoidal components that share a fundamental frequency $f_0 = 50$ Hz. Hence, the powers of these components can just be summed to obtain the average power of signal $x(t)$. For each component, its power equals the component's amplitude squared, divided by 2. The four components have frequencies $f_1 = 50$ Hz, $f_2 = 100$ Hz, $f_3 = 200$ Hz, $f_4 = 500$ Hz. From now on, we use subscripts 50, 100, 200 and 500 to refer to elements concerning the respective individual components. The powers of the four components are:

$P_x|_{f=50} = \frac{2^2}{2} = 2$ W, $P_x|_{f=100} = \frac{4^2}{2} = 8$ W, $P_x|_{f=200} = \frac{2^2}{2} = 2$ W, and $P_x|_{f=500} = \frac{1^2}{2} = \frac{1}{2}$ W

The equalizer illustration in Exercise e.2-36 shows that the equalizer only passes signal components of frequencies 50 Hz, 100 Hz, 250 Hz and 500 Hz. This means that zero power of the sinusoidal component of 200 Hz will be passed to the output.
Also given is that $P_y = 20$ W. Hence, we know that:

$$P_y = G_{50}P_x|_{f=50} + G_{100}P_x|_{f=100} + G_{500}P_x|_{f=500} = 2G_{50} + 8G_{100} + \tfrac{1}{2}G_{500} = 20 \text{ W},$$

where G_f correponds to the (equalizer) gain for the sinusoidal component with frequency f.
The equalizer gain at 100 Hz is set at 0 dB, which means that the input power at 100 Hz is passed 1 : 1 to the output, hence $G_{100} = 1$.
In combination with the statement that the equalizer gain at 50 Hz is set 10 dB higher than the equalizer gain at 500 Hz, i.e., $G_{500} = 0.1G_{50}$, we obtain the following equation:

$$2G_{50} + 8\cdot 1 + \tfrac{1}{2}\cdot 0.1G_{50} = 20, \quad \text{or} \quad 2.05G_{50} = 12$$

Hence: $G_{50} = \frac{12}{2.05}$ and $G_{500} = 0.1\cdot\frac{12}{2.05}$

Converting to dB, we obtain: $G_{50} = 10\log_{10}\left(\frac{12}{2.05}\right) \approx 7.67$ dB

$$G_{500} = 10\log_{10}\left(0.1\cdot\frac{12}{2.05}\right) \approx -2.33 \text{ dB}$$

II

Continuous time

3

Real Fourier series

Exercises

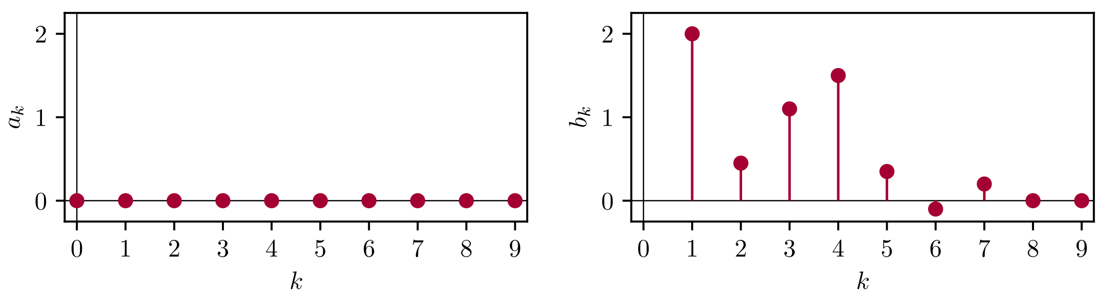
e|**3.1** Rationale and definition

Obtain the real Fourier series coefficients of the following signals without doing any integration: *e*.3-1
(a) $2\cos(3\pi t) + 4\sin(5\pi t)$ (b) $\cos^2(50\pi t)$ (c) $\sin(\pi t)\cos^2(20\pi t)$

Consider the three periodic signals from Exercise *e*.2-14 on page 12. For each of these signals: *e*.3-2
- (1) Obtain its real Fourier series coefficients.
- (2) Relate these coefficients to the phasors as obtained in Exercise *e*.2-14.

e|**3.5** Effects of symmetry

Consider a periodic signal $x(t)$, of which the 10 real Fourier series coefficients a_k and b_k are shown below. *e*.3-3



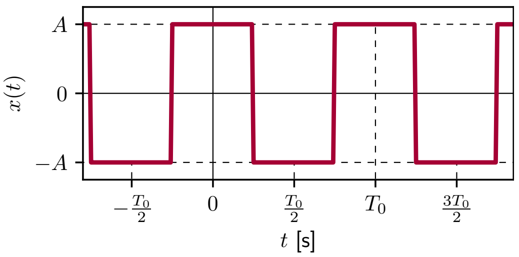
Determine for each of the following statements whether the statement is true or false:
- (I) $x(t)$ is an odd signal
- (II) The average of $x(t)$ is zero
- (III) $x(t)$ has half-wave odd symmetry

e.3-4 Obtain the real Fourier series coefficients of the following signals:

(a) Square wave, defined by:

$$x(t) = \begin{cases} A & \text{for } -\frac{T_0}{4} < t \leq \frac{T_0}{4} \\ -A & \text{for } -\frac{T_0}{2} < t \leq -\frac{T_0}{4} \text{ and } \frac{T_0}{4} < t \leq \frac{T_0}{2} \end{cases}$$

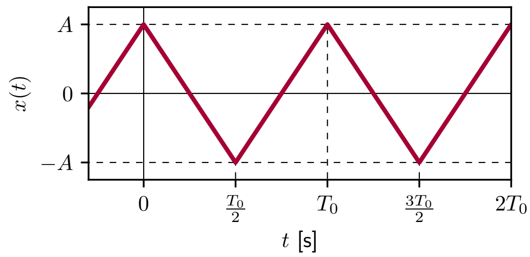
and periodically extended outside this interval (i.e., $x(t) = x(t + T_0)$ for all t).



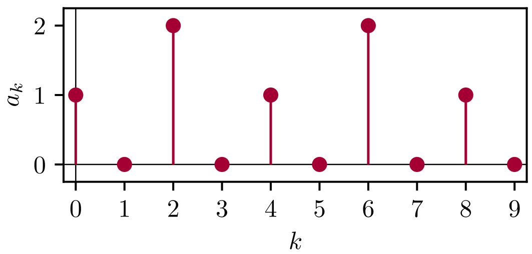
(b) Triangular wave, defined by:

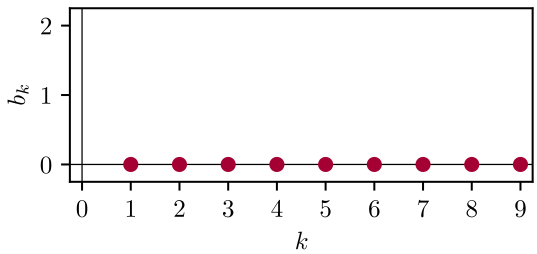
$$x(t) = \begin{cases} A + \frac{4A}{T_0}t & \text{for } -\frac{T_0}{2} \leq t \leq 0 \\ A - \frac{4A}{T_0}t & \text{for } 0 \leq t \leq \frac{T_0}{2} \end{cases}$$

and periodically extended outside this interval (i.e., $x(t) = x(t + T_0)$ for all t).



e.3-5 Consider a periodic signal $x(t)$, of which the 10 real Fourier series coefficients a_k and b_k are shown below.





Determine for each of the following statements whether the statement is true or false:

(I) $x(t)$ is an even signal

(II) The average of $x(t)$ is 0.7

(III) $x(t)$ has half-wave odd symmetry

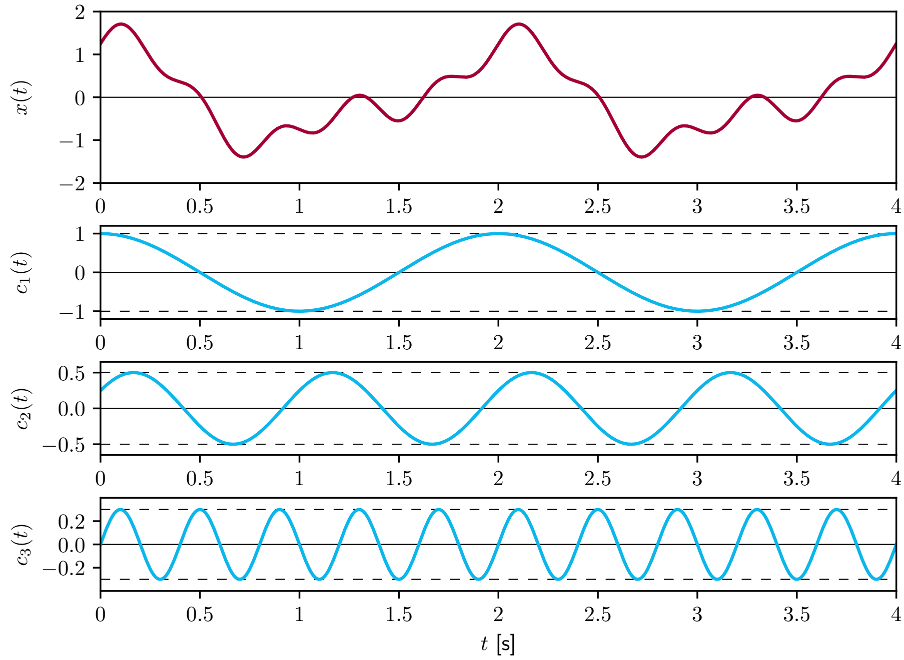
e.3-6 Show that the real, or trigonometric, Fourier series of a real, even signal consists only of cosine terms, and that of an odd signal consists only of sine terms.

e.3-7 Show that for a signal with half-wave odd symmetry, all even-indexed coefficients ($k = 0, 2, 4, 6, \ldots$) of the real Fourier series are zero.

e|3.6 Single-sided spectrum

e.3-8

We construct a periodic signal $x(t)$ using three summed components $c_1(t)$, $c_2(t)$ and $c_3(t)$. Below, an integer number of periods of signal $x(t)$ (top plot, in burgundy) and of its components (bottom plots, in blue) are illustrated.



From this illustration, retrieve the real Fourier series coefficients of signal $x(t)$.

Select the correct answer from the options below.

A. $a_2 = 1$, $a_4 = \frac{1}{4}$, $b_4 = \frac{1}{4}\sqrt{3}$, $b_{10} = \frac{3}{10}$

B. $a_1 = 1$, $a_2 = \frac{1}{4}$, $b_2 = \frac{1}{4}\sqrt{3}$, $b_5 = \frac{3}{10}$

C. $a_1 = 1$, $a_2 = -\frac{1}{4}$, $b_2 = \frac{1}{4}\sqrt{3}$, $b_5 = \frac{3}{10}$

D. $a_1 = 1$, $a_2 = -\frac{1}{4}$, $a_5 = \frac{3}{10}$, $b_2 = \frac{1}{4}\sqrt{3}$

e.3-9

Consider the three periodic signals from Exercise *e*.2-14 on page 12. For each of these signals:

(1) From its real Fourier series coefficients obtained in Exercise *e*.3-2, compute the $\{A_k, \theta_k\}$ values of all harmonics as used in the alternative definition of the real Fourier series, (3.14).

(2) Relate these values to the magnitudes and phase angles of the phasors as obtained in Exercise *e*.2-14.

e | 3.c Computer exercises

e.3-10 In Section 3.6.2 an alternative expression for the real Fourier series was given, which uses only the cosine function:

$$x(t) = a_0 + \sum_{k=1}^{k=\infty} A_k \cos(k\omega_0 t + \theta_k) \tag{3.14}$$

In this exercise we investigate why, for an arbitrary kth harmonic with *positive* Fourier series coefficients $a_k > 0$ and $b_k > 0$, phase angle θ_k is *negative*:

$$\theta_k = \arctan\left(-\frac{b_k}{a_k}\right) = -\arctan\left(\frac{b_k}{a_k}\right) \tag{3.13}$$

(a) Construct a signal $x(t)$ using the real Fourier series with fundamental frequency $f_0 = 2$ Hz, coefficients $a_1 = 2$ and $b_1 = 1$, and all other coefficients equal to zero. Plot its components $a_1 \cos(\omega_0 t)$ and $b_1 \sin(\omega_0 t)$ and the constructed signal against time.

(b) Comment on the shape and phase of $x(t)$.

(c) Derive an expression for the time shift t_{0_k} associated with the phase θ_k of an arbitrary kth harmonic, with arbitrary non-zero $a_k \in \mathbb{R}$ and $b_k \in \mathbb{R}$.

e.3-11 (a) Construct a signal $x(t)$ using the real Fourier series with fundamental frequency $f_0 = 1$ Hz, coefficients $b_1 = 1.0$ and $b_2 = 0.4$, and all other coefficients equal to zero. Plot the constructed signal against time.

(b) Comment on the shape of the constructed signal, comparing it to a pure sine with unit amplitude and frequency $f_0 = 1$ Hz.
Hint: plot the pure sine in the same figure as the constructed signal.

(c) Now construct the signal $x(t)$ using the real Fourier series with the same fundamental frequency and coefficients as in (a), but with in addition $b_3 = 0.1$. Again, plot the constructed signal against time and comment on its shape.

Solutions

s|3.1 Rationale and definition

s.3-1

(a) $2\cos(3\pi t) + 4\sin(5\pi t) = 2\cos\big(3(1\pi)t\big) + 4\sin\big(5(1\pi)t\big)$
The average is zero, hence $a_0 = 0$ (see Section 3.2 to learn why a_0 is simply the signal's average). The highest common frequency, and therefore the fundamental frequency ω_0, equals π rad/s ($f_0 = 0.5$ Hz). There are two sinusoidal components, a cosine 3 times the fundamental frequency and a sine 5 times the fundamental frequency. Hence, $a_3 = 2$ and $b_5 = 4$. All other real Fourier series coefficients are zero.

(b) $\cos^2(50\pi t) = \frac{1}{2}\big(1 + \cos(100\pi t)\big) = \frac{1}{2} + \frac{1}{2}\cos(100\pi t)$
The average is $\frac{1}{2}$, hence $a_0 = \frac{1}{2}$. The fundamental frequency ω_0 equals 100π rad/s ($f_0 = 50$ Hz). There is only one sinusoidal component, a cosine, hence $a_1 = \frac{1}{2}$. All other real Fourier series coefficients are zero.

(c) $\sin(\pi t)\cos^2(20\pi t) = \sin(\pi t)\,\frac{1}{2}\big(1 + \cos(40\pi t)\big) = \frac{1}{2}\sin(\pi t) + \frac{1}{2}\sin(\pi t)\cos(40\pi t)$
Using $\sin u \cos v = \frac{1}{2}\big(\sin(u - v) + \sin(u + v)\big)$, (A.10):
$\frac{1}{2}\sin(\pi t) + \frac{1}{4}\sin(-39\pi t) + \frac{1}{4}\sin(41\pi t) = \frac{1}{2}\sin\big(1(1\pi)t\big) - \frac{1}{4}\sin\big(39(1\pi)t\big) + \frac{1}{4}\sin\big(41(1\pi)t\big)$
The average is zero, hence $a_0 = 0$. The highest common frequency, and therefore the fundamental frequency ω_0, equals π rad/s ($f_0 = 0.5$ Hz). There are three sinusoidal components, all sines, with frequencies 1, 39 and 41 times the fundamental frequency. Hence, $b_1 = \frac{1}{2}$, $b_{39} = -\frac{1}{4}$, $b_{41} = \frac{1}{4}$. All other real Fourier series coefficients are zero.

s.3-2

In this exercise, we use the trigonometric identities $\sin(u \pm v) = \sin u \cos v \pm \cos u \sin v$, (A.6), and $\cos(u \pm v) = \cos u \cos v \mp \sin u \sin v$, (A.7), to get rid of the phase terms.

(a)
$$\begin{aligned}
x_a(t) &= 4\cos\Big(2\pi t + \frac{\pi}{4}\Big) - 2\sin\Big(14\pi t - \frac{\pi}{2}\Big)\\
&= 4\cos(2\pi t)\cos\Big(\frac{\pi}{4}\Big) - 4\sin(2\pi t)\sin\Big(\frac{\pi}{4}\Big)\\
&\qquad - 2\sin(14\pi t)\cos\Big(-\frac{\pi}{2}\Big) - 2\cos(14\pi t)\sin\Big(-\frac{\pi}{2}\Big)\\
&= 4\cos(2\pi t)\,\frac{1}{2}\sqrt{2} - 4\sin(2\pi t)\,\frac{1}{2}\sqrt{2} - 2\sin(14\pi t)\cdot 0 - 2\cos(14\pi t)\,(-1)\\
&= 2\sqrt{2}\cos(2\pi t) - 2\sqrt{2}\sin(2\pi t) + 2\cos(14\pi t)
\end{aligned}$$
The fundamental frequency f_0 is 1 Hz.
We obtain: $a_1 = 2\sqrt{2}$, $b_1 = -2\sqrt{2}$ and $a_7 = 2$
Relating these to the phasors as obtained in Exercise *e*.2-14(a) (see Solution *s*.2-14(a)), we see that the real part of phasor $\vec{X}_k$ corresponds to the a_k coefficient (here $k = 1, 7$), and the imaginary part of phasor $\vec{X}_k$ to the $-b_k$ coefficient.

(b)
$$\begin{aligned}
x_b(t) &= 1 - 2\sin\Big(2\pi t + \frac{\pi}{4}\Big) + 4\cos\Big(5\pi t - \frac{\pi}{2}\Big)\\
&= 1 - 2\sin(2\pi t)\cos\Big(\frac{\pi}{4}\Big) - 2\cos(2\pi t)\sin\Big(\frac{\pi}{4}\Big)\\
&\qquad + 4\cos(5\pi t)\cos\Big(-\frac{\pi}{2}\Big) - 4\sin(5\pi t)\sin\Big(-\frac{\pi}{2}\Big)\\
&= 1 - 2\sin(2\pi t)\,\frac{1}{2}\sqrt{2} - 2\cos(2\pi t)\,\frac{1}{2}\sqrt{2} + 4\cos(5\pi t)\cdot 0 - 4\sin(5\pi t)\,(-1)\\
&= 1 - \sqrt{2}\sin(2\pi t) - \sqrt{2}\cos(2\pi t) + 4\sin(5\pi t)
\end{aligned}$$
The fundamental frequency f_0 is 0.5 Hz.

We obtain: $a_0 = 1, \quad a_2 = -\sqrt{2}, \quad b_2 = -\sqrt{2}$ and $b_5 = 4$

Relating these to the phasors as obtained in Exercise e.2-14(b) (see Solution s.2-14(b)), we find the same relationship as in (a).

(c)
$$\begin{aligned} x_c(t) &= -2 + 4\sin(4\pi t) + 3\cos\left(4\pi t + \tfrac{\pi}{4}\right) \\ &= -2 + 4\sin(4\pi t) + 3\cos(4\pi t)\cos\left(\tfrac{\pi}{4}\right) - 3\sin(4\pi t)\sin\left(\tfrac{\pi}{4}\right) \\ &= -2 + 4\sin(4\pi t) + 3\cos(4\pi t)\,\tfrac{1}{2}\sqrt{2} - 3\sin(4\pi t)\,\tfrac{1}{2}\sqrt{2} \\ &= -2 + \left(4 - \tfrac{3}{2}\sqrt{2}\right)\sin(4\pi t) + \tfrac{3}{2}\sqrt{2}\cos(4\pi t) \end{aligned}$$

The fundamental frequency f_0 is 2 Hz.

We obtain: $a_0 = -2, \quad a_1 = \frac{3}{2}\sqrt{2}$ and $b_1 = 4 - \frac{3}{2}\sqrt{2}$

Relating these to the phasors as obtained in Exercise e.2-14(c) (see Solution s.2-14(c)), we find the same relationship as in (a).

Concluding, we see that the real Fourier series coefficients and the phasor description are related:
$$\vec{X}_0 = a_0 \quad \text{and} \quad \vec{X}_k = a_k - jb_k, \quad k > 0$$

s|3.5 Effects of symmetry

s.3-3

(I) True, as for an odd signal, all a_k's equal zero, as is the case for $x(t)$.

(II) True, as a signal's average is expressed by a_0, which is zero for $x(t)$.

(III) False, as for signals with half-wave odd symmetry, all even-indexed coefficients are zero, which is clearly not the case for $x(t)$.

s.3-4

(a) From the illustration of the signal, it can directly be seen that the signal is real and even, so the Fourier series consists of only cosine terms. Hence, all b_k's equal zero.

$$\begin{aligned} a_0 &= \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\,\mathrm{d}t = \frac{1}{T_0}\left(\int_{-\frac{T_0}{2}}^{-\frac{T_0}{4}} (-A)\,\mathrm{d}t + \int_{-\frac{T_0}{4}}^{\frac{T_0}{4}} A\,\mathrm{d}t + \int_{\frac{T_0}{4}}^{\frac{T_0}{2}} (-A)\,\mathrm{d}t\right) \\ &= \frac{A}{T_0}\left(\left[-t\right]_{-T_0/2}^{-T_0/4} + \left[t\right]_{-T_0/4}^{T_0/4} + \left[-t\right]_{T_0/4}^{T_0/2}\right) = 0 \end{aligned}$$

$a_0 = 0$ could also be directly deduced from the illustration of $x(t)$, as it clearly shows that $x(t)$ has zero mean.

$$\begin{aligned} a_k &= \frac{2}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\cos(k\omega_0 t)\,\mathrm{d}t \\ &= \frac{2}{T_0}\left(\int_{-\frac{T_0}{2}}^{-\frac{T_0}{4}} -A\cos(k\omega_0 t)\,\mathrm{d}t + \int_{-\frac{T_0}{4}}^{\frac{T_0}{4}} A\cos(k\omega_0 t)\,\mathrm{d}t + \int_{\frac{T_0}{4}}^{\frac{T_0}{2}} -A\cos(k\omega_0 t)\,\mathrm{d}t\right) \\ &= \left[\frac{-2A}{k\omega_0 T_0}\sin(k\omega_0 t)\right]_{-\frac{T_0}{2}}^{-\frac{T_0}{4}} + \left[\frac{2A}{k\omega_0 T_0}\sin(k\omega_0 t)\right]_{-\frac{T_0}{4}}^{\frac{T_0}{4}} + \left[\frac{-2A}{k\omega_0 T_0}\sin(k\omega_0 t)\right]_{\frac{T_0}{4}}^{\frac{T_0}{2}} \\ &= \frac{2A}{k\omega_0 T_0}\Big(\sin\left(k\omega_0\tfrac{T_0}{4}\right) - \sin\left(k\omega_0\tfrac{T_0}{2}\right) + \sin\left(k\omega_0\tfrac{T_0}{4}\right) + \sin\left(k\omega_0\tfrac{T_0}{4}\right) \\ &\qquad - \sin\left(k\omega_0\tfrac{T_0}{2}\right) + \sin\left(k\omega_0\tfrac{T_0}{4}\right)\Big) \\ &= \frac{2A}{k\omega_0 T_0}\left(4\sin\left(k\omega_0\tfrac{T_0}{4}\right) - 2\sin\left(k\omega_0\tfrac{T_0}{2}\right)\right) \end{aligned}$$

This expression for a_k can be simplified by noting that $\omega_0 = \frac{2\pi}{T_0}$:

$$a_k = \frac{2A}{k\pi}\left(2\sin\left(k\frac{\pi}{2}\right) - \sin(k\pi)\right)$$

The second term on the right-hand side is zero for all indices k and can be discarded. For even indices k (including $k=0$), the first term on the right-hand side also equals zero, so $a_k = 0$ for k even. Furthermore:

For $k = 1,\ 5,\ 9, \ldots\ :\quad \sin\left(k\frac{\pi}{2}\right) = 1$ For $k = 3,\ 7,\ 11, \ldots\ :\quad \sin\left(k\frac{\pi}{2}\right) = -1$

We only use the positive indices k as per the definition of the real Fourier series, (3.1).

Concluding: $$a_k = \begin{cases} \frac{4A}{k\pi} & \text{for } k = 1, 5, 9, \ldots \\ -\frac{4A}{k\pi} & \text{for } k = 3, 7, 11, \ldots \\ 0 & \text{for } k \text{ even (including } k = 0) \end{cases}$$

We see that we only need the odd-indexed coefficients, all even-indexed coefficients (including $k = 0$) are zero. This is a consequence of this signal being half-wave odd symmetric (see Exercise e.3-7). Remember that a signal with this symmetry property is defined to have zero mean (see Table 2.1), i.e., $a_0 = 0$.

Compare the real Fourier series coefficients of this even square wave with those computed in Example 3.1 for an odd-symmetric square wave. Note that the even square wave is obtained by shifting the odd-symmetric square wave in Example 3.1 to the left by a quarter of a wavelength $\left(\frac{T_0}{4}\right)$. Also note that the effect of this time shift on the real Fourier series coefficients was already discussed in Example 3.2, and that the coefficients of this even square wave were also already computed in Example 3.5, but then using the coefficients of the odd-symmetric square wave from Example 3.1 as the starting point.

(b) From the illustration of the triangular wave signal, it can directly be seen that, similar to the square wave signal in (a), this signal is real and even, so the real Fourier series consists of only cosine terms. Hence, all b_k's equal zero.

$$\begin{aligned} a_0 &= \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\,\mathrm{d}t = \frac{1}{T_0}\left(\int_{-\frac{T_0}{2}}^{0}\left(A + \frac{4A}{T_0}t\right)\mathrm{d}t + \int_{0}^{\frac{T_0}{2}}\left(A - \frac{4A}{T_0}t\right)\mathrm{d}t\right) \\ &= \frac{1}{T_0}\left(\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} A\,\mathrm{d}t + \frac{4A}{T_0}\int_{-\frac{T_0}{2}}^{0} t\,\mathrm{d}t - \frac{4A}{T_0}\int_{0}^{\frac{T_0}{2}} t\,\mathrm{d}t\right) \\ &= \frac{A}{T_0}\left([t]_{-\frac{T_0}{2}}^{\frac{T_0}{2}} + \left[\frac{2}{T_0}t^2\right]_{-\frac{T_0}{2}}^{0} + \left[-\frac{2}{T_0}t^2\right]_{0}^{\frac{T_0}{2}}\right) = \frac{A}{T_0}\left(\left(\frac{T_0}{2} + \frac{T_0}{2}\right) + \left(0 - \frac{T_0}{2}\right) + \left(-\frac{T_0}{2} - 0\right)\right) = 0 \end{aligned}$$

$a_0 = 0$ could also be directly deduced from the illustration of $x(t)$, as it clearly shows that $x(t)$ has zero mean.

$$\begin{aligned} a_k &= \frac{2}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\cos(k\omega_0 t)\ \mathrm{d}t \\ &= \frac{2}{T_0}\left(\int_{-\frac{T_0}{2}}^{0}\left(A + \frac{4A}{T_0}t\right)\cos(k\omega_0 t)\ \mathrm{d}t + \int_{0}^{\frac{T_0}{2}}\left(A - \frac{4A}{T_0}t\right)\cos(k\omega_0 t)\ \mathrm{d}t\right) \\ &= \frac{2A}{T_0}\left(\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}}\cos(k\omega_0 t)\ \mathrm{d}t + \frac{4}{T_0}\int_{-\frac{T_0}{2}}^{0} t\cos(k\omega_0 t)\ \mathrm{d}t - \frac{4}{T_0}\int_{0}^{\frac{T_0}{2}} t\cos(k\omega_0 t)\ \mathrm{d}t\right) \end{aligned}$$

The first integral on the right-hand side equals zero for all indices k, as we integrate a cosine that fits exactly k times in the period T_0 of integration.

For the remaining two integrals on the right-hand side we use the partial integration rule that states that $\int_a^b \frac{\mathrm{d}f(x)}{\mathrm{d}x} g(x)\,\mathrm{d}x = [f(x)g(x)]_a^b - \int_a^b f(x)\frac{\mathrm{d}g(x)}{\mathrm{d}x}\,\mathrm{d}x$:

define $\frac{\mathrm{d}f(t)}{\mathrm{d}t} = \cos(k\omega_0 t)$ and $g(t) = t$, from which it follows that $f(t) = \frac{1}{k\omega_0}\sin(k\omega_0 t)$ and $\frac{\mathrm{d}g(t)}{\mathrm{d}t} = 1$, and we obtain:

$$a_k = \frac{8A}{T_0^2}\left(\left[\frac{t}{k\omega_0}\sin(k\omega_0 t)\right]_{-\frac{T_0}{2}}^{0} - \frac{1}{k\omega_0}\int_{-\frac{T_0}{2}}^{0}\sin(k\omega_0 t)\ \mathrm{d}t \right.$$
$$\left. - \left[\frac{t}{k\omega_0}\sin(k\omega_0 t)\right]_{0}^{\frac{T_0}{2}} + \frac{1}{k\omega_0}\int_{0}^{\frac{T_0}{2}}\sin(k\omega_0 t)\ \mathrm{d}t\right)$$
$$= \frac{8A}{T_0^2}\left(\left(0 + \frac{T_0}{2k\omega_0}\sin\left(-k\omega_0\frac{T_0}{2}\right)\right) + \left[\frac{1}{k^2\omega_0^2}\cos(k\omega_0 t)\right]_{-\frac{T_0}{2}}^{0}\right.$$
$$\left. - \left(\frac{T_0}{2k\omega_0}\sin\left(k\omega_0\frac{T_0}{2}\right) - 0\right) + \left[-\frac{1}{k^2\omega_0^2}\cos(k\omega_0 t)\right]_{0}^{\frac{T_0}{2}}\right)$$
$$= \frac{8A}{T_0^2}\left(\frac{T_0}{2k\omega_0}\sin\left(-k\omega_0\frac{T_0}{2}\right) + \frac{1}{k^2\omega_0^2} - \frac{1}{k^2\omega_0^2}\cos\left(-k\omega_0\frac{T_0}{2}\right)\right.$$
$$\left. - \frac{T_0}{2k\omega_0}\sin\left(k\omega_0\frac{T_0}{2}\right) - \frac{1}{k^2\omega_0^2}\cos\left(k\omega_0\frac{T_0}{2}\right) + \frac{1}{k^2\omega_0^2}\right)$$

Using $\omega_0 = \frac{2\pi}{T_0}$:

$$a_k = \frac{8A}{T_0^2}\left(\frac{\pi}{k\omega_0^2}\sin(-k\pi) + \frac{1}{k^2\omega_0^2} - \frac{1}{k^2\omega_0^2}\cos(-k\pi) - \frac{\pi}{k\omega_0^2}\sin(k\pi) - \frac{1}{k^2\omega_0^2}\cos(k\pi) + \frac{1}{k^2\omega_0^2}\right)$$
$$= \frac{8A}{T_0^2}\left(\frac{2}{k^2\omega_0^2} - \frac{1}{k^2\omega_0^2}(\cos(k\pi) + \cos(-k\pi))\right)$$

For even indices k, both $\cos(k\pi)$ and $\cos(-k\pi)$ equal 1. For odd indices k, both $\cos(k\pi)$ and $\cos(-k\pi)$ equal -1. Hence:
For even indices k:

$$a_k = \frac{8A}{T_0^2}\left(\frac{2}{k^2\omega_0^2} - \frac{1}{k^2\omega_0^2}(1+1)\right) = \frac{8A}{T_0^2}\left(\frac{2}{k^2\omega_0^2} - \frac{2}{k^2\omega_0^2}\right) = 0$$

For odd indices k:

$$a_k = \frac{8A}{T_0^2}\left(\frac{2}{k^2\omega_0^2} - \frac{1}{k^2\omega_0^2}(-1+(-1))\right) = \frac{8A}{T_0^2}\left(\frac{2}{k^2\omega_0^2} + \frac{2}{k^2\omega_0^2}\right) = \frac{32A}{T_0^2 k^2\omega_0^2} = \frac{32A}{k^2(2\pi)^2} = \frac{8A}{k^2\pi^2}$$

Concluding: $$a_k = \begin{cases} \frac{8A}{k^2\pi^2} & \text{for } k \text{ odd} \\ 0 & \text{for } k \text{ even (including } k=0) \end{cases}$$

We see that, similar to the square wave signal in (a), we only need the odd-indexed coefficients. All even-indexed coefficients (including $k = 0$) are zero, a consequence of this signal being half-wave odd symmetric (see Exercise e.3-7). Hence, all even-indexed coefficients being zero could also be directly deduced from the illustration of $x(t)$, as it clearly shows that $x(t)$ has half-wave odd symmetry. Remember that, by definition, a signal with half-wave odd symmetry has zero mean (see Table 2.1), i.e., $a_0 = 0$.

s.3-5

(I) True, as for an even signal, all b_k's equal zero, as is the case for $x(t)$.
(II) False, as a signal's average is expressed by a_0, which is 1 for $x(t)$.
(III) False, as for signals with half-wave odd symmetry, all even-indexed coefficients are zero, not, as is the case for $x(t)$, all odd-indexed coefficients.

s.3-6

Given the definition of the real, or trigonometric, Fourier series in (3.1), to show that the trigonometric Fourier series of a real, even signal consists only of cosine terms, we must show that for such a signal, all b_k's are zero.

Remember: an *even* function is a function where $\quad x(t) = x(-t) \quad$ (see Table 2.1).

Take (3.5) for coefficients b_k as a starting point: $b_k = \frac{2}{T_0}\int_{T_0} x(t)\sin(k\omega_0 t)\,\mathrm{d}t \quad$ with $\; k \in \mathbb{N}^+$

For all b_k's to be zero, we must show that $\int_{T_0} x(t)\sin(k\omega_0 t)\,\mathrm{d}t = 0$ for all indices k.

For a function to be even, its behavior for positive and negative time t is important, i.e., time relative to $t = 0$, so we take the integral over period T_0 to be symmetric about $t = 0$:

$$\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\sin(k\omega_0 t)\,\mathrm{d}t = \int_{-\frac{T_0}{2}}^{0} x(t)\sin(k\omega_0 t)\,\mathrm{d}t + \int_{0}^{\frac{T_0}{2}} x(t)\sin(k\omega_0 t)\,\mathrm{d}t$$

In the first integral on the right-hand side, we substitute $t = -\sigma$, then $\mathrm{d}t = \mathrm{d}(-\sigma)$; then for $t = -\frac{T_0}{2}$, we get $\sigma = \frac{T_0}{2}$, and for $t = 0$, we get $\sigma = 0$:

$$\int_{\sigma=\frac{T_0}{2}}^{0} x(-\sigma)\sin(-k\omega_0\sigma)\,\mathrm{d}(-\sigma) + \int_{t=0}^{\frac{T_0}{2}} x(t)\sin(k\omega_0 t)\,\mathrm{d}t$$

Now, look at the integrand in the first integral: (1) $\sin(-k\omega_0\sigma) = -\sin(k\omega_0\sigma)$ (a sine is an odd function), (2) reversing the interval limits causes an additional minus sign, and (3) $\mathrm{d}(-\sigma) = -\,\mathrm{d}\sigma$, so we get three minus signs in the first integral. Also, for an even function, $x(-\sigma) = x(\sigma)$. Together, this results in:

$$\underset{\substack{\uparrow\\ \text{interval}\\ \text{limit}\\ \text{reversal}}}{-}\;\overset{\substack{\text{sine}\\ \text{component}\\ \downarrow}}{-}\;\underset{\substack{\uparrow\\ (-\,\mathrm{d}\sigma)}}{-}\int_{\sigma=0}^{\frac{T_0}{2}} x(\sigma)\sin(k\omega_0\sigma)\,\mathrm{d}\sigma + \int_{t=0}^{\frac{T_0}{2}} x(t)\sin(k\omega_0 t)\,\mathrm{d}t$$

Then, we simply substitute $\sigma = t$ in the first integral,[1] then $\mathrm{d}\sigma = \mathrm{d}t$; for $\sigma = 0$, we get $t = 0$, and for $\sigma = \frac{T_0}{2}$, we get $t = \frac{T_0}{2}$. We obtain:

$$-\int_{t=0}^{\frac{T_0}{2}} x(t)\sin(k\omega_0 t)\,\mathrm{d}t + \int_{t=0}^{\frac{T_0}{2}} x(t)\sin(k\omega_0 t)\,\mathrm{d}t = 0 \qquad \textbf{qed}$$

Hence, for a real, even signal, all b_k's are zero, and its real Fourier series consists only of cosine terms.

Equivalently, to show that the trigonometric Fourier series of an odd signal consists only of sine terms, we must show that for such a signal, all a_k's are zero.

Remember: an *odd* function is a function where $\quad x(t) = -x(-t) \quad$ (see Table 2.1).

Take (3.4) for coefficients a_k as a starting point: $a_k = \frac{2}{T_0}\int_{T_0} x(t)\cos(k\omega_0 t)\,\mathrm{d}t \quad$ with $\; k \in \mathbb{N}^+$

For all a_k's to be zero, we must show that $\int_{T_0} x(t)\cos(k\omega_0 t)\,\mathrm{d}t = 0$ for all indices k.

For a function to be odd, its behavior for positive and negative time t is important, i.e., time relative to $t = 0$, so we take the integral over period T_0 to be symmetric about $t = 0$:

$$\int_{T_0} x(t)\cos(k\omega_0 t)\,\mathrm{d}t = \int_{-\frac{T_0}{2}}^{0} x(t)\cos(k\omega_0 t)\,\mathrm{d}t + \int_{0}^{\frac{T_0}{2}} x(t)\cos(k\omega_0 t)\,\mathrm{d}t$$

We perform the same substitution in the first integral on the right-hand side as we did for b_k:

$$\int_{\sigma=\frac{T_0}{2}}^{0} x(-\sigma)\cos(-k\omega_0\sigma)\,\mathrm{d}(-\sigma) + \int_{t=0}^{\frac{T_0}{2}} x(t)\cos(k\omega_0 t)\,\mathrm{d}t$$

Now, look at the integrand in the first integral: (1) for an odd function, $x(-\sigma) = -x(\sigma)$, (2) reversing the interval limits causes an additional minus sign, and (3) $\mathrm{d}(-\sigma) = -\,\mathrm{d}\sigma$, so we get three minus signs in the first integral. Also, $\cos(-k\omega_0\sigma) = \cos(k\omega_0\sigma)$ (a cosine is an even function). Together, this results in:

$$\underset{\substack{\uparrow\\ \text{interval limit reversal}}}{-}\ \overset{\substack{\text{odd function } x\text{: } (-x(\sigma))\\ \downarrow}}{-}\ \underset{\substack{\uparrow\\ (-\,\mathrm{d}\sigma)}}{-}\int_{\sigma=0}^{\frac{T_0}{2}} x(\sigma)\cos(k\omega_0\sigma)\ \mathrm{d}\sigma + \int_{t=0}^{\frac{T_0}{2}} x(t)\cos(k\omega_0 t)\ \mathrm{d}t$$

Then, we simply substitute $\sigma = t$ in the first integral again, and we obtain:

$$-\int_{t=0}^{\frac{T_0}{2}} x(t)\cos(k\omega_0 t)\ \mathrm{d}t + \int_{t=0}^{\frac{T_0}{2}} x(t)\cos(k\omega_0 t)\ \mathrm{d}t = 0 \qquad \textbf{qed}$$

Hence, for an odd signal, all a_k's are zero, and its Fourier series consists only of sine terms.

[1]This step might raise the question: Why do we first substitute $t = -\sigma$ in the step above, and here $\sigma = t$ again? Well, the answer is simple. These substitutions have *nothing* to do with each other. We could also substitute $\sigma = a$ in the second step, and call a the running variable. The 'name of' or 'symbol used' for the running variable in the integral is arbitrary, so why would we not take t? We are usually dealing with time signals, and the second integral also uses t. We could also substitute $t = \sigma$ in the second integral and then we would get an equality in σ, it does not matter at all. Note that a substitution is just a mathematical trick to (in this case) change the integral limits to make them more convenient. We can use this trick as many times as we like.

S.3-7

We must show that, for a signal with half-wave odd symmetry, all a_k's and b_k's for k even ($k = 0, 2, 4, 6, \ldots$) are zero, including a_0. By definition (see Table 2.1), a half-wave odd signal has zero mean, and since a_0 is simply the average value of a signal, it follows that $a_0 = 0$ for such signals. Let us start with a_k:

$$a_k = \frac{2}{T_0}\int_{T_0} x(t)\cos(k\omega_0 t)\ \mathrm{d}t = \frac{2}{T_0}\left(\int_{-\frac{T_0}{2}}^{0} x(t)\cos(k\omega_0 t)\ \mathrm{d}t + \int_{0}^{\frac{T_0}{2}} x(t)\cos(k\omega_0 t)\ \mathrm{d}t\right)$$

Remember: a *half-wave odd* function is a function where $x\left(t \pm \frac{T_0}{2}\right) = -x(t)$ (see Table 2.1).

In the first integral on the right-hand side, we substitute $t = \sigma - \frac{T_0}{2}$, then $\mathrm{d}t = \mathrm{d}\sigma$; for $t = -\frac{T_0}{2}$, we get $\sigma = 0$, and for $t = 0$, we get $\sigma = \frac{T_0}{2}$. When also using $\omega_0 = 2\pi f_0 = \frac{2\pi}{T_0}$:

$$a_k = \frac{2}{T_0}\left(\int_{\sigma=0}^{\frac{T_0}{2}} \underbrace{x\left(\sigma - \frac{T_0}{2}\right)}_{=\,-x(\sigma)\quad \text{for half-wave odd functions}}\cos\left(k\frac{2\pi}{T_0}\left(\sigma - \frac{T_0}{2}\right)\right)\ \mathrm{d}\sigma + \int_{t=0}^{\frac{T_0}{2}} x(t)\cos(k\omega_0 t)\ \mathrm{d}t\right)$$

Using trigonometric identity $\cos(u-v) = \cos u\cos v + \sin u \sin v$, (A.7):

$$\begin{aligned}
\cos\left(k\frac{2\pi}{T_0}\left(\sigma - \frac{T_0}{2}\right)\right) &= \cos\left(k\frac{2\pi}{T_0}\sigma\right)\cos\left(k\frac{2\pi}{T_0}\frac{T_0}{2}\right) + \sin\left(k\frac{2\pi}{T_0}\sigma\right)\sin\left(k\frac{2\pi}{T_0}\frac{T_0}{2}\right)\\
&= \cos\left(k\frac{2\pi}{T_0}\sigma\right)\cos(k\pi) + \sin\left(k\frac{2\pi}{T_0}\sigma\right)\sin(k\pi)\\
&= \cos(k\omega_0\sigma)\cos(k\pi) \qquad \text{as } \sin(k\pi) \text{ is } 0 \text{ for all indices } k.
\end{aligned}$$

$$a_k = \frac{2}{T_0}\left(-\int_{\sigma=0}^{\frac{T_0}{2}} x(\sigma)\cos(k\omega_0\sigma)\cos(k\pi)\ \mathrm{d}\sigma + \int_{t=0}^{\frac{T_0}{2}} x(t)\cos(k\omega_0 t)\ \mathrm{d}t\right)$$

Then, we simply substitute $\sigma = t$ in the first integral, and we obtain:

$$a_k = \frac{2}{T_0}\left(1 - \cos(k\pi)\right)\int_{0}^{\frac{T_0}{2}} x(t)\cos(k\omega_0 t)\ \mathrm{d}t$$

For even indices k (including $k = 0$), the term $\left(1 - \cos(k\pi)\right)$ equals zero. For odd indices k, $\left(1 - \cos(k\pi)\right)$ equals 2. Hence:

$$a_k = \begin{cases} 0 & \text{for } k \text{ even (including } k = 0)\\ \frac{4}{T_0}\int_{0}^{\frac{T_0}{2}} x(t)\cos(k\omega_0 t)\ \mathrm{d}t & \text{for } k \text{ odd}\end{cases} \qquad \textbf{qed}$$

Hence, for a signal with half-wave odd symmetry, all even-indexed a_k coefficients (including $k = 0$) are zero. The same procedure can be followed for b_k.

s|3.6 Single-sided spectrum

s.3-8

The correct answer is B.

From the illustration, $c_1(t)$ has exactly two oscillations in 4 s, $c_2(t)$ exactly four oscillations, and $c_3(t)$ exactly ten oscillations. Thus, the frequencies of the three components are:

$$f_1 = \frac{1}{T_1} = \frac{1}{2} = 0.5 \text{ Hz}, \quad f_2 = \frac{1}{T_2} = \frac{1}{1} = 1 \text{ Hz}, \quad \text{and} \quad f_3 = \frac{1}{T_3} = \frac{1}{0.4} = 2.5 \text{ Hz}$$

Hence, the fundamental frequency $f_0 = 0.5$ Hz.

$c_1(t)$ is a *cosine* signal with amplitude 1 and frequency f_1 equal to $1f_0$: $a_1 = 1$

$c_3(t)$ is a *sine* signal with amplitude 0.3 and frequency f_3 equal to $5f_0$: $b_5 = \frac{3}{10}$

$c_2(t)$ is a sinusoidal signal with amplitude 0.5 and frequency f_2 equal to $2f_0$ and phase θ

Hence, $c_2(t) = A_k \cos(k\omega_0 t + \theta_k) = a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t)$ with $k = 2$ and $A_2 = 0.5$

From the illustration, we can see that at $t = 0$ s: $c_2(0) = 0.25$

From this, we can deduce that at $t = 0$ s: $0.25 = 0.5\cos\big(2(2\pi)(0.5)(0) + \theta_2\big) = 0.5\cos\theta_2$

Hence, $\theta_2 = -\frac{\pi}{3}$, as we can also see that $c_2(t)$ is a cosine signal shifted to the *right*, and thus θ_2 is negative. Using $a_k = A_k \cos\theta_k$ and $b_k = -A_k \sin\theta_k$, (3.11), we obtain:

$$a_2 = 0.5\cos\left(-\frac{\pi}{3}\right) = \frac{1}{4} \quad \text{and} \quad b_2 = -0.5\sin\left(-\frac{\pi}{3}\right) = \frac{1}{4}\sqrt{3}$$

s.3-9

(a) $x_a(t) = 4\cos\left(2\pi t + \frac{\pi}{4}\right) - 2\sin\left(14\pi t - \frac{\pi}{2}\right)$ has fundamental frequency f_0 is 1 Hz.

As the first component is already written as $A_1\cos(\omega_0 t + \theta_1)$, we can directly obtain:

$$A_1 = 4 \quad \text{and} \quad \theta_1 = \frac{\pi}{4}$$

Writing the second component as a cosine yields:

$$-2\sin\left(14\pi t - \frac{\pi}{2}\right) = -2\cos\left(14\pi t - \frac{\pi}{2} - \frac{\pi}{2}\right) = -2\cos(14\pi t - \pi) = 2\cos(14\pi t)$$

Hence, we obtain: $A_7 = 2$ and $\theta_7 = 0$

Alternatively, in Solution *s*.3-2(a), we obtained: $a_1 = 2\sqrt{2}$, $b_1 = -2\sqrt{2}$ and $a_7 = 2$

Then, using (3.12) and (3.13), we also obtain:

$$A_1 = \sqrt{a_1^2 + b_1^2} = \sqrt{(2\sqrt{2})^2 + (-2\sqrt{2})^2} = 4$$

$$\theta_1 = \arctan\left(-\frac{b_1}{a_1}\right) = \arctan\left(-\frac{-2\sqrt{2}}{2\sqrt{2}}\right) = \arctan(1) = \frac{\pi}{4}$$

$$A_7 = \sqrt{a_7^2 + b_7^2} = \sqrt{2^2 + 0^2} = 2 \quad \text{and} \quad \theta_7 = \arctan\left(-\frac{b_7}{a_7}\right) = \arctan\left(-\frac{0}{2}\right) = 0$$

These values correspond exactly to the magnitude and phase of the phasors $\vec{X}_k$ obtained in Exercise *e*.2-14(a), see Solution *s*.2-14(a).

(b) $x_b(t) = 1 - 2\sin\left(2\pi t + \frac{\pi}{4}\right) + 4\cos\left(5\pi t - \frac{\pi}{2}\right)$ has fundamental frequency f_0 is 0.5 Hz.

We directly obtain: $A_0 = 1$ and $\theta_0 = 0$, $A_5 = 4$ and $\theta_5 = -\frac{\pi}{2}$

Writing the second component as a cosine yields:

$$-2\sin\left(2\pi t + \frac{\pi}{4}\right) = -2\cos\left(2\pi t + \frac{\pi}{4} - \frac{\pi}{2}\right) = -2\cos\left(2\pi t - \frac{\pi}{4}\right) = 2\cos\left(2\pi t + \frac{3\pi}{4}\right)$$

Hence, we obtain: $A_2 = 2$ and $\theta_2 = \frac{3\pi}{4}$

Alternatively, in Solution *s*.3-2(b), we obtained: $a_2 = -\sqrt{2}$, $b_2 = -\sqrt{2}$ and $b_5 = 4$

Then, using (3.12) and (3.13), we also obtain:

$$A_2 = \sqrt{a_2^2 + b_2^2} = \sqrt{(-\sqrt{2})^2 + (-\sqrt{2})^2} = 2$$

$$\theta_2 = \arctan\left(-\frac{b_2}{a_2}\right) = \arctan\left(-\frac{-\sqrt{2}}{-\sqrt{2}}\right) = \frac{3\pi}{4}$$ [1]

$$A_5 = \sqrt{a_5^2 + b_5^2} = \sqrt{0^2 + 4^2} = 4$$

$$\theta_5 = \arctan\left(-\frac{b_5}{a_5}\right) = \arctan\left(-\frac{4}{0}\right) = \arctan(-\infty) = -\frac{\pi}{2}$$

Again, these values correspond exactly to the magnitude and phase of the phasors $\vec{X}_k$ obtained in Exercise *e*.2-14(b), see Solution *s*.2-14(b).

(c) $x_c(t) = -2 + 4\sin(4\pi t) + 3\cos\left(4\pi t + \frac{\pi}{4}\right)$ has fundamental frequency f_0 is 2 Hz.
We directly obtain: $A_0 = 2$ and $\theta_0 = \pi$
In Solution *s*.3-2(c), we obtained: $a_1 = \frac{3}{2}\sqrt{2}$ and $b_1 = 4 - \frac{3}{2}\sqrt{2}$
Then, using (3.12) and (3.13), we obtain:

$$A_1 = \sqrt{a_1^2 + b_1^2} = \sqrt{(\tfrac{3}{2}\sqrt{2})^2 + (4 - \tfrac{3}{2}\sqrt{2})^2} = \sqrt{25 - 12\sqrt{2}}$$

$$\theta_1 = \arctan\left(-\frac{b_1}{a_1}\right) = \arctan\left(-\frac{4 - \frac{3}{2}\sqrt{2}}{\frac{3}{2}\sqrt{2}}\right) = \arctan\left(-\frac{4}{3}\sqrt{2} + 1\right)$$

Again, these values correspond exactly to the magnitude and phase of the phasors $\vec{X}_k$ in Exercise *e*.2-14(c), see Solution *s*.2-14(c).

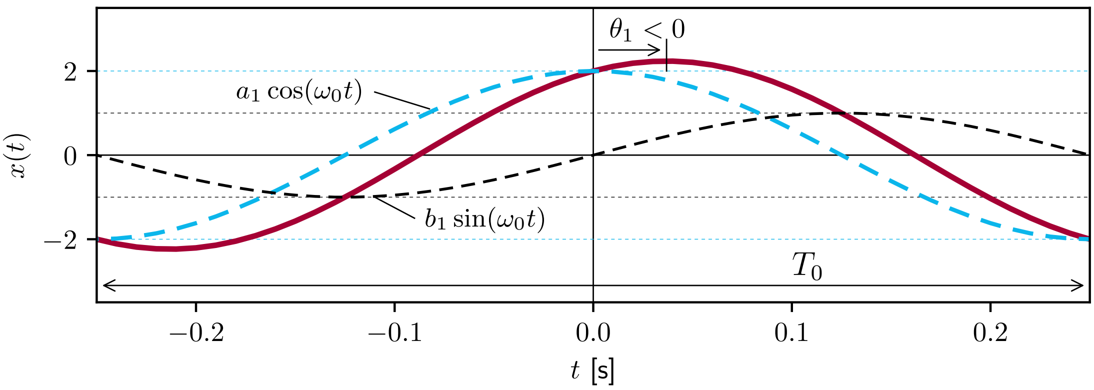
Concluding, we see that $A_k = |\vec{X}_k|$ and $\theta_k = \angle\vec{X}_k$. When using the alternative formulation of the real Fourier series, (3.14), the resulting amplitudes A_k and phases θ_k correspond exactly to, respectively, the magnitudes $|\vec{X}_k|$ and phases $\angle\vec{X}_k = \theta_k$ of the phasors $\vec{X}_k$.

[1]Note that the four-quadrant arctangent is used, both `numpy.atan2()` and `numpy.arctan2()` in Python.

s | 3.c Computer exercises

s.3-10

(a) Signal $x(t)$ (in burgundy) and its components are shown below.



(b) In the real Fourier series, all harmonics can be written as a cosine function; the kth harmonic having amplitude A_k and phase θ_k, (3.14). In this exercise, signal $x(t)$ equals the first harmonic, $k = 1$, and with $a_1 = 2$ and $b_1 = 1$ the result is a cosine shifted to the right of $t = 0$ s: its phase θ_1 is negative. The harmonic (shown in burgundy) lags behind a zero-phase cosine (compare $x(t)$ with its cosine component $a_1\cos(\omega_0 t)$ shown in dashed blue). With $a_1 = 2$ and $b_1 = 1$ we get $\theta_1 = -\arctan\left(\frac{1}{2}\right) = -0.46$ rad.

(c) For the kth harmonic of a real Fourier series we have:

$$A_k\cos(k\omega_0 t + \theta_k) = A_k\cos\left(k\omega_0\left(t + \frac{\theta_k}{k\omega_0}\right)\right) = A_k\cos\left(k\omega_0(t + t_{0_k})\right)$$

Substituting $\omega_0 = 2\pi f_0$, we obtain: $t_{0_k} = \frac{1}{k2\pi f_0}\theta_k$

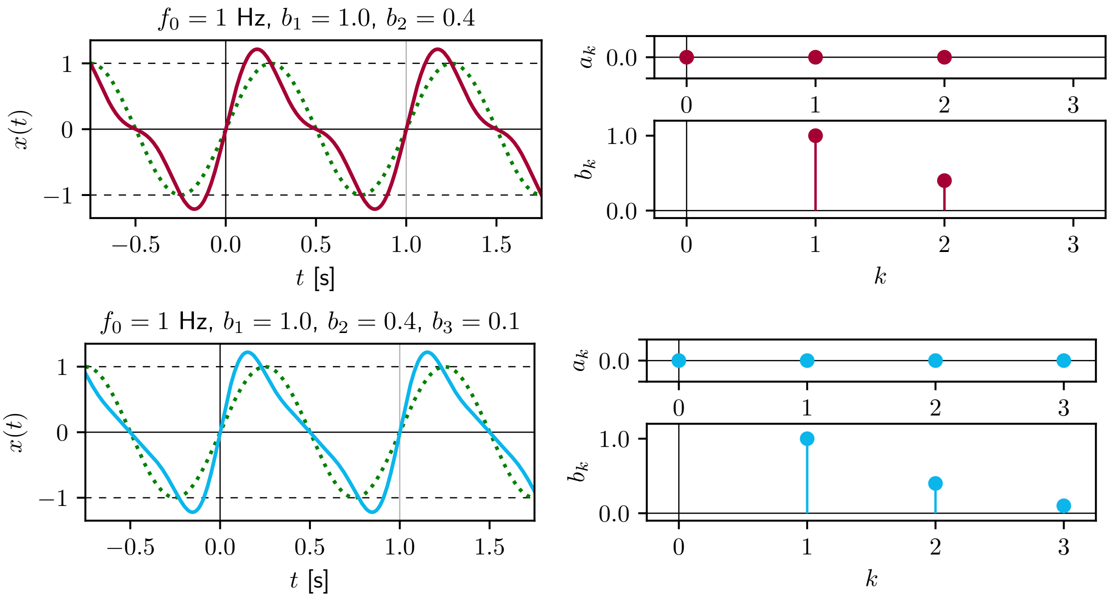
With $\theta_1 = -0.46$ rad (for $k = 1$) and $f_0 = 2$ Hz we get $t_{0_1} = -0.0369$ s.

Clearly, when $a_k > 0$ and $b_k > 0$, $\theta_k < 0$, then $t_{0_k} < 0$: the kth harmonic lags behind a zero-phase cosine.

Basic Python code:

```
import numpy as np
from matplotlib import pyplot as plt

burgundy = [165/255,0/255,52/255]        # burgundy color

# signal x(t): real Fourier series consisting of only first harmonic (k=1)
f0 = 2                                   # fundamental frequency [Hz]
a1 = 2.0                                 # cosine coefficient (amplitude)
b1 = 1.0                                 #   sine coefficient (amplitude)
                                         # all other coefficients zero
t = np.linspace(-0.35, 0.35, 71)  # time steps 0.01 s from -0.35 to 0.35 s
cost = a1 * np.cos(2*np.pi*f0*t)         # cosine component
sint = b1 * np.sin(2*np.pi*f0*t )        #   sine component
xt   = cost + sint                       # signal x(t)
t0   = np.atan2(b1,a1) / (2*np.pi*f0)    # time shift of first harmonic [s]

# plot x(t) and its components
plt.figure()
plt.plot(t, xt, color=burgundy, linewidth=2)
plt.plot(t, cost, color='b', linestyle='--', dashes=(5,3), linewidth=1.5)
plt.plot(t, sint, color='k', linestyle='--', dashes=(5,3), linewidth=1.0)
plt.xlabel('$t$ [s]'); plt.ylabel('$x(t)$')
```

(a)-(c) The result of this Fourier series is a *skewed* sine, shown below in burgundy at the top (for $b_1 = 1.0$, $b_2 = 0.4$), and in blue at the bottom (for $b_1 = 1.0$, $b_2 = 0.4$, $b_3 = 0.1$), with the pure $f_0 = 1$ Hz sine for reference in dotted green. The first peak of the sine has been pushed back to the left, and the second (negative) peak to the right. This behavior often occurs in spectral analysis practice. A perfect sine will hardly emerge in physical reality. A 'skewed' sine as shown here, is often encountered. In order to construct such a signal, sines with higher frequencies are needed, in addition to the sine of fundamental frequency f_0. These higher frequencies are integer multiples of the fundamental frequency. These sines, in this case with frequencies $2f_0$ and $3f_0$, are referred to as *higher harmonics*. S.3-11



Basic Python code:

```
import numpy as np
from matplotlib import pyplot as plt

burgundy = [165/255,0/255,52/255]  # burgundy color

# real Fourier series, with just one additional component (k=2), and
# two additional components (k=2, k=3), to show effect of higher harmonics
b1 = 1          # k=1; b1 = 1      pure sine at fundamental freq f0 = 1 Hz
b2 = 0.4        # k=2; b2 = 0.4
b3 = 0.1        # k=3; b3 = 0.1
                # all other coefficients zero
t  = np.linspace(-0.75, 1.75, 251) #time steps 0.01 s from -0.75 to 1.75 s
x1 = b1 * np.sin(2*np.pi*1*t)
x2 = b2 * np.sin(2*np.pi*2*t)
x3 = b3 * np.sin(2*np.pi*3*t)

# real Fourier series with one additional component (k=2)
fig = plt.figure()
fig.add_axes([0.05, 0.7, 0.4, 0.25])
plt.plot(t, x1, color='g', linestyle=':')   # pure sinusoid: signal
plt.plot(t, (x1+x2), color=burgundy)        # real FS with k=2: time trace
plt.xlabel('$t$ [s]'); plt.ylabel('$x(t)$')
# coefficients
fig.add_axes([0.55, 0.9, 0.4, 0.05])
k  = np.array([0, 1, 2])
ak = np.zeros(3)                            # all a_k's equal zero
plt.stem(k, ak, basefmt=' ', linefmt=burgundy, markerfmt=burgundy)
plt.ylabel('$a_k$')
fig.add_axes([0.55, 0.7, 0.4, 0.15])
k  = np.array([1, 2])
bk = np.array([b1, b2])                     # b_1, b_2 non-zero
plt.stem(k, bk, basefmt=' ', linefmt=burgundy, markerfmt=burgundy)
plt.xlabel('$k$'); plt.ylabel('$b_k$')

# real Fourier series with two additional components (k=2 and k=3)
fig.add_axes([0.05, 0.3, 0.4, 0.25])
plt.plot(t, x1, color='g', linestyle=':')   # pure sinusoid: signal
plt.plot(t, (x1+x2+x3), color='b')  # real FS with k=2 and k=3: time trace
plt.xlabel('$t$ [s]'); plt.ylabel('$x(t)$')
# coefficients
fig.add_axes([0.55, 0.5, 0.4, 0.05])
k  = np.array([0, 1, 2, 3])
ak = np.zeros(4)                            # all a_k's equal zero
plt.stem(k, ak, basefmt=' ', linefmt='b', markerfmt='b')
plt.ylabel('$a_k$')
fig.add_axes([0.55, 0.3, 0.4, 0.15])
k  = np.array([1, 2, 3])
bk = np.array([b1, b2, b3])                 # b_1, b_2, b_3 non-zero
plt.stem(k, bk, basefmt=' ', linefmt='b', markerfmt='b')
plt.xlabel('$k$'); plt.ylabel('$b_k$')
```

4

Complex exponential Fourier series

Exercises

e|4.1 Derivation of the complex exponential Fourier series

e.4-1

Prove the following relationships, which relate the coefficients of the complex exponential Fourier series to those of the real, or trigonometric, Fourier series:

$$\begin{aligned} X_0 &= a_0 \\ \left.\begin{aligned} X_k &= \tfrac{1}{2}(a_k - jb_k) \\ X_{-k} &= \tfrac{1}{2}(a_k + jb_k) \end{aligned}\right\} & \text{ for } k \in \mathbb{N}^+ \end{aligned} \tag{4.4}$$

e|4.2 Interpretation, polar form

e.4-2

Consider the three periodic signals from Exercise *e*.2-14 on page 12, for which the real Fourier series coefficients were obtained in Exercise *e*.3-2. For each of these signals:

(1) Obtain its complex exponential Fourier series coefficients.

(2) Relate these coefficients to the complex conjugate phasor pairs as obtained in Exercise *e*.2-15.

e.4-3

In this exercise we continue with Exercise *e*.3-10 on page 46. Consider the following signal:

$$x(t) = 2\cos(4\pi t) + \sin(4\pi t)$$

(a) Express $x(t)$ using the complex exponential Fourier series, (4.2).

(b) Sketch the positions of the complex exponential Fourier series conjugates at $t = 0$ s as vectors in the complex plane. In your sketch, indicate the direction in which these conjugates are rotating when time progresses, $t \geq 0$.

(c) Compute the time t' at which the two conjugates will cross the real axis for the first time after $t = 0$ s.

(d) Compare your answer in (c) with the answer to Exercise *e*.3-10(c) on page 54 and explain the result.

e.4-4 The complex exponential Fourier series of a signal over an interval $0 \le t \le T_0$ is

$$x(t) = \sum_{k=-\infty}^{\infty} \frac{1}{1+j\pi k} e^{j(5\pi k t/2)}$$

(a) Determine the numerical value of period T_0.

(b) What is the average value of $x(t)$ over the interval $[0, T_0]$?

(c) Determine the amplitude and phase of the fifth harmonic component.

(d) Express the fifth harmonic term in the Fourier series in terms of a cosine.

e|4.3 Effects of symmetry

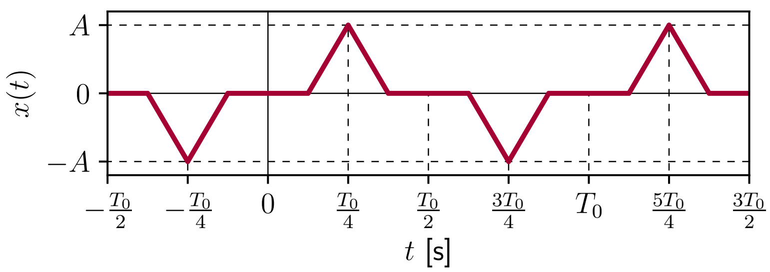
e.4-5 Prove the following symmetry properties for the complex exponential Fourier series coefficients:

(a) X_k is real and an even function of k if $x(t)$ is real and even.

(b) X_k is imaginary and odd if $x(t)$ is real and odd.

(c) $X_k = 0$ for even k if $x(t)$ has half-wave odd symmetry.

e.4-6 Consider the periodic signal illustrated below.



(a) What is the value of a_0 in the trigonometric Fourier series?

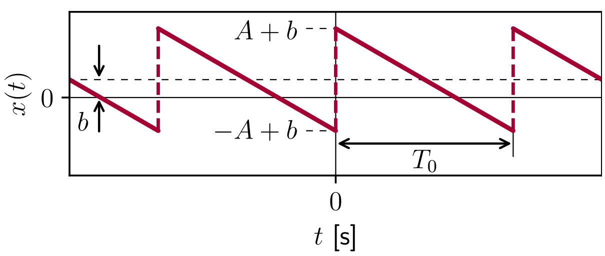
(b) What are the values of the a_k coefficients in the trigonometric Fourier series?

(c) Are the coefficients X_k in the complex exponential Fourier series real, imaginary, or complex-valued?

(d) What are the values of the even-indexed coefficients X_k in the complex exponential Fourier series?

Justify your answers.

e.4-7 Consider the periodic signal $x(t)$ with period T_0 shown on the right. The signal can be expressed as a complex exponential Fourier series with coefficients X_k.



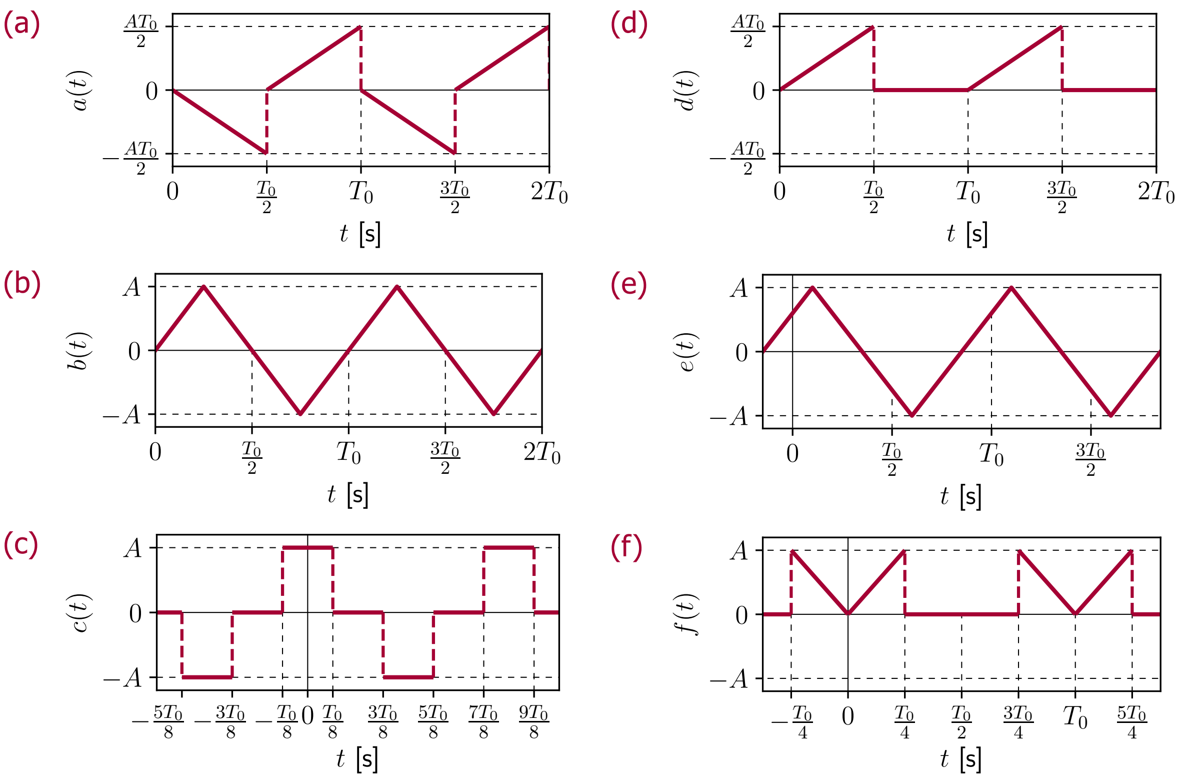
Determine for each of the two statements below whether the statement is true or false:

(I) $X_0 = b$

(II) If $b \neq 0$, all coefficients X_k (except $k = 0$) will be purely imaginary

e.4-8

Consider the six periodic signals illustrated below and their complex exponential Fourier series. For each of these signals, determine whether each of the following properties is true or false: (1) purely real coefficients, (2) purely imaginary coefficients, (3) complex coefficients, (4) even-indexed coefficients are zero, (5) $X_0 = 0$. If true, check the appropriate box in the table at the bottom of this exercise.



	Signal					
Property	(a)	(b)	(c)	(d)	(e)	(f)
(1) Purely real coefficients						
(2) Purely imaginary coefficients						
(3) Complex coefficients						
(4) Even-indexed coefficients are zero						
(5) $X_0 = 0$						

e|**4.5** Examples

e.4-9

Obtain the complex exponential Fourier series coefficients of the following signals without doing any integration. Also give the fundamental frequency f_0 of each signal.

(a) $x_a(t) = 1 - 2\sin(5\pi t) + 3\cos\left(10\pi t + \frac{\pi}{3}\right)$

(b) $x_b(t) = 5\cos^2(30\pi t)\sin(15\pi t)$

(c) $x_c(t) = 3\sin^3(35\pi t) + 4\cos(20\pi t)$

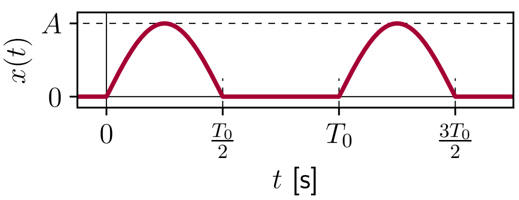
(d) $x_d(t) = 2\sin^2(70\pi t)\cos^2(30\pi t) + 7\sin(20\pi t)\cos(5\pi t)$

e.4-10 Obtain the complex exponential Fourier series coefficients of the following signals:
(*Tip*: check whether obtained coefficients are consistent with the signal's symmetry properties.)

(a) Half-rectified sine wave, defined by:

$$x(t) = \begin{cases} A\sin(\omega_0 t) & \text{for } 0 \le t \le \frac{T_0}{2} \\ 0 & \text{for } \frac{T_0}{2} \le t \le T_0 \end{cases}$$

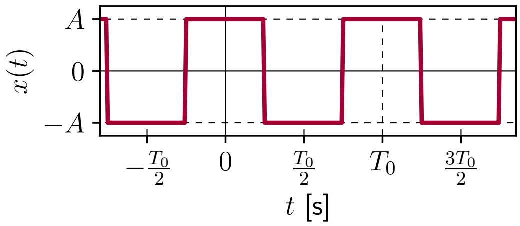
and periodically extended outside this interval (i.e., $x(t) = x(t+T_0)$ for all t), where $A > 0$ is a real constant.



(c) Square wave, defined by:

$$x(t) = \begin{cases} A & \text{for } -\frac{T_0}{4} < t \le \frac{T_0}{4} \\ -A & \text{for } -\frac{T_0}{2} < t \le -\frac{T_0}{4} \text{ and} \\ & \quad \frac{T_0}{4} < t \le \frac{T_0}{2} \end{cases}$$

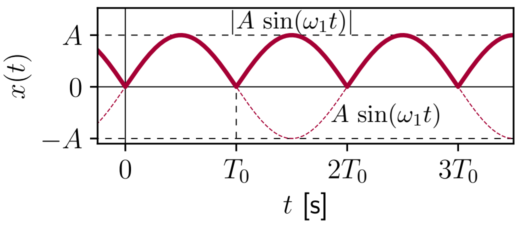
and periodically extended outside this interval (i.e., $x(t) = x(t+T_0)$ for all t), where $A > 0$ is a real constant.



(b) Full-rectified sine wave, defined by:

$$x(t) = \left|A\sin(\omega_1 t)\right|$$

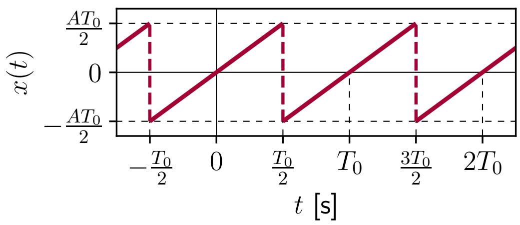
where $A > 0$ is a real constant.



(d) Sawtooth waveform, defined by:

$$x(t) = At \quad \text{for } -\frac{T_0}{2} \le t < \frac{T_0}{2}$$

and periodically extended outside this interval (i.e., $x(t) = x(t+T_0)$ for all t), where $A > 0$ is a real constant.

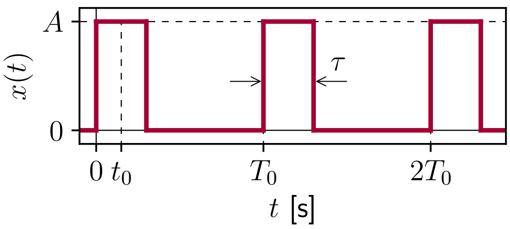


e.4-11 Obtain the complex exponential Fourier series coefficients of the following signals:

(a) Pulse train signal, defined by:

$$x(t) = \sum_{n=-\infty}^{\infty} A\Pi\left(\frac{t - t_0 - nT_0}{\tau}\right) \quad \text{with } \tau < T_0$$

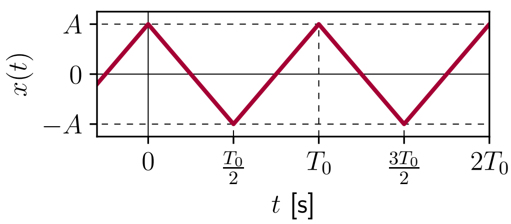
where $A > 0$ is a real constant.



(b) Triangular wave, defined by:

$$x(t) = \begin{cases} A + \frac{4A}{T_0}t & \text{for } -\frac{T_0}{2} \le t \le 0 \\ A - \frac{4A}{T_0}t & \text{for } 0 \le t \le \frac{T_0}{2} \end{cases}$$

and periodically extended outside this interval (i.e., $x(t) = x(t+T_0)$ for all t), where $A > 0$ is a real constant.



e|4.6 Two Fourier series theorems

e.4-12 Prove the complex exponential Fourier series time shift theorem (4.13).

e.4-13 Prove the complex exponential Fourier series differentiation theorem (4.15).

Hint: suppose that differentiation of a periodic signal $x(t)$ results in a signal $x'(t) = \frac{\mathrm{d}x(t)}{\mathrm{d}t}$ that has a complex exponential Fourier series described by coefficients X'_k. Integrate by parts the expression:

$$X'_k = \frac{1}{T_0}\int_{T_0} \frac{\mathrm{d}x(t)}{\mathrm{d}t} e^{-jk\omega_0 t}\,\mathrm{d}t$$

to show that:

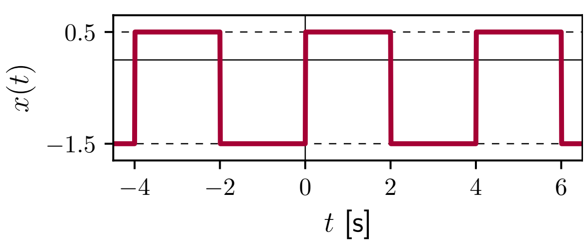
$$X'_k = (jk\omega_0)X_k$$

That is, the complex exponential Fourier series coefficients of the signal $x'(t) = \frac{\mathrm{d}x(t)}{\mathrm{d}t}$ are related to the complex exponential Fourier series coefficients of signal $x(t)$ through multiplication by $jk\omega_0$, as described by the differentiation theorem (4.15).

e.4-14 Starting from the complex exponential Fourier series coefficients of the triangular wave obtained in Exercise e.4-11(b), obtain the complex exponential Fourier series coefficients of an odd-symmetry square wave.

e.4-15 Consider the periodic signal $x(t)$ shown below on the right. The complex exponential Fourier series coefficients of this signal, X_k, are defined by:

$$X_k = \begin{cases} -j\frac{2}{k\pi} & \text{for } k \text{ odd} \\ -\frac{1}{2} & \text{for } k = 0 \\ 0 & \text{for } k \text{ even (except } k = 0) \end{cases}$$



Signal $y(t)$ is defined as the second time derivative of $x(t)$: $\quad y(t) = \frac{\mathrm{d}^2}{\mathrm{d}^2 t}x(t)$

Compute the complex exponential Fourier series coefficients Y_k of signal $y(t)$ for k equal to $-2, -1, 0, 1$ and 2.

e.4-16 Given the complex exponential Fourier series coefficients X_k of a signal $x(t)$, obtain the coefficients Y_k of a signal $y(t)$ in terms of X_k, if the two signals are related by:

(a) $y(t) = x(t-\tau)$, where τ is a constant;

(b) $y(t) = x(t)\,e^{j2\pi f_0 t}$, where f_0 is a constant.

e|4.7 Parseval: complex exponential Fourier series

e.4-17 Show that the average power of a periodic signal $x(t)$ can be computed with Parseval's theorem, using the complex exponential Fourier series coefficients, as defined in (4.16):

$$P = X_0^2 + 2\sum_{k=1}^{\infty} |X_k|^2$$

Solutions

S|4.1 Derivation of the complex exponential Fourier series

s.4-1

Note that the relationships (4.4) between the coefficients of the complex exponential Fourier series and those of the real Fourier series are also derived in Section 4.1. However, it is always good to practice these derivations yourself, hence, this exercise.

The real, or trigonometric, Fourier series are defined as:

$$x(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(k\omega_0 t) + \sum_{k=1}^{\infty} b_k \sin(k\omega_0 t) \qquad (3.1)$$

The complex exponential Fourier series are defined as: $$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{jk\omega_0 t} \qquad (4.2)$$

Let us start off with substituting the complex exponential forms of $\cos(k\omega_0 t)$ and $\sin(k\omega_0 t)$ into the expression for the trigonometric Fourier series, i.e., substitute $\cos(k\omega_0 t) = \frac{1}{2}\left(e^{-jk\omega_0 t} + e^{jk\omega_0 t}\right)$ (C.2) and $\sin(k\omega_0 t) = j\frac{1}{2}\left(e^{-jk\omega_0 t} - e^{jk\omega_0 t}\right)$ (C.3)

into (3.1): $$x(t) = a_0 + \sum_{k=1}^{\infty} a_k \frac{1}{2}(e^{-jk\omega_0 t} + e^{jk\omega_0 t}) + \sum_{k=1}^{\infty} b_k \frac{j}{2}(e^{-jk\omega_0 t} - e^{jk\omega_0 t})$$

Collecting the terms associated with the positive and negative exponents, we get:

$$x(t) = a_0 + \sum_{k=1}^{\infty} \frac{1}{2}(a_k - jb_k)e^{jk\omega_0 t} + \sum_{k=1}^{\infty} \frac{1}{2}(a_k + jb_k)e^{-jk\omega_0 t}$$

Re-defining the index in the second summation yields:

$$x(t) = a_0 + \sum_{k=1}^{\infty} \frac{1}{2}(a_k - jb_k)e^{jk\omega_0 t} + \sum_{k=-1}^{-\infty} \frac{1}{2}(a_{-k} + jb_{-k})e^{jk\omega_0 t}$$

When comparing this expression to (4.2) and recalling that $e^{j0\omega_0 t} = 1$, it becomes clear that:

$X_0 = a_0$ and $X_k = \frac{1}{2}(a_k - jb_k)$ and $X_{-k} = \frac{1}{2}(a_k + jb_k)$ for $k \in \mathbb{N}^+$ **qed**

Note that in the line immediately above, k is always defined to be positive.

S|4.2 Interpretation, polar form

s.4-2

(a) $x_a(t) = 4\cos\left(2\pi t + \frac{\pi}{4}\right) - 2\sin\left(14\pi t - \frac{\pi}{2}\right)$

In Solution *s*.3-2(a), we obtained: $a_1 = 2\sqrt{2}$, $b_1 = -2\sqrt{2}$ and $a_7 = 2$

Hence, using (4.4), we obtain for the complex exponential Fourier series coefficients:

$$X_1 = \tfrac{1}{2}(a_1 - jb_1) = \tfrac{1}{2}\left(2\sqrt{2} - j(-2\sqrt{2}\,)\right) = \sqrt{2} + j\sqrt{2} = \sqrt{2}\,(1+j)$$

$$X_{-1} = \tfrac{1}{2}(a_1 + jb_1) = \tfrac{1}{2}\left(2\sqrt{2} + j(-2\sqrt{2}\,)\right) = \sqrt{2} - j\sqrt{2} = \sqrt{2}\,(1-j)$$

An easier way to obtain X_{-1} is of course: $X_{-1} = X_1^*$.

$$X_7 = \tfrac{1}{2}(a_7 - jb_7) = \tfrac{1}{2}(2 - 0j) = 1 \quad \text{and} \quad X_{-7} = X_7^* = 1$$

Relating these to the complex conjugate phasor pairs $\{\frac{1}{2}\vec{X}_k, \frac{1}{2}\vec{X}_{-k}\}$ obtained in Exercise *e*.2-15 (see Solution *s*.2-15(a)), we see that the complex exponential Fourier series coefficients are exactly the same!

$$X_k = \tfrac{1}{2}\vec{X}_k \quad \text{and} \quad X_{-k} = \tfrac{1}{2}\vec{X}_{-k}$$

(b) $x_b(t) = 1 - 2\sin\left(2\pi t + \frac{\pi}{4}\right) + 4\cos\left(5\pi t - \frac{\pi}{2}\right)$

In Solution *s*.3-2(b), we obtained: $a_0 = 1$, $a_2 = -\sqrt{2}$, $b_2 = -\sqrt{2}$ and $b_5 = 4$

Hence, using (4.4), we obtain for the complex exponential Fourier series coefficients:

$X_0 = a_0 = 1$

$X_2 = \frac{1}{2}(a_2 - jb_2) = -\frac{1}{2}\sqrt{2} + j\frac{1}{2}\sqrt{2} = -\frac{1}{2}\sqrt{2}\,(1-j)$ and $X_{-2} = X_2^* = -\frac{1}{2}\sqrt{2}\,(1+j)$

$X_5 = \frac{1}{2}(a_5 - jb_5) = \frac{1}{2}(0 - 4j) = -2j$ and $X_{-5} = X_5^* = 2j$

Relating these to the complex conjugate phasor pairs obtained in Exercise *e*.2-15 (see Solution *s*.2-15(b)), we find the same relationship as in (a), and that $X_0 = \vec{X}_0$.

(c) $x_c(t) = -2 + 4\sin(4\pi t) + 3\cos\left(4\pi t + \frac{\pi}{4}\right)$

In Solution *s*.3-2(c), we obtained: $a_0 = -2,\quad a_1 = \frac{3}{2}\sqrt{2}$ and $b_1 = 4 - \frac{3}{2}\sqrt{2}$

Hence, using (4.4), we obtain for the complex exponential Fourier series coefficients:

$X_0 = a_0 = -2$

$X_1 = \frac{1}{2}(a_1 - jb_1) = \frac{3}{4}\sqrt{2} + j(\frac{3}{4}\sqrt{2} - 2)$ and $X_{-1} = X_1^* = \frac{3}{4}\sqrt{2} - j(\frac{3}{4}\sqrt{2} - 2)$

Relating these to the complex conjugate phasor pairs obtained in Exercise *e*.2-15 (see Solution *s*.2-15(c)), we again find the same relationships as in (a) and (b).

Concluding this exercise, we see that the complex exponential Fourier series coefficients X_k *are* the complex conjugate phasor pairs:

$$X_k = \tfrac{1}{2}\vec{X}_k \quad \text{for} \quad k \in \mathbb{Z}\setminus\{0\} \quad \text{and} \quad X_0 = \vec{X}_0$$

They scale the magnitude and set the phase (at $t = 0$ s) of the complex exponential basis functions corresponding to the kth harmonic.

s.4-3

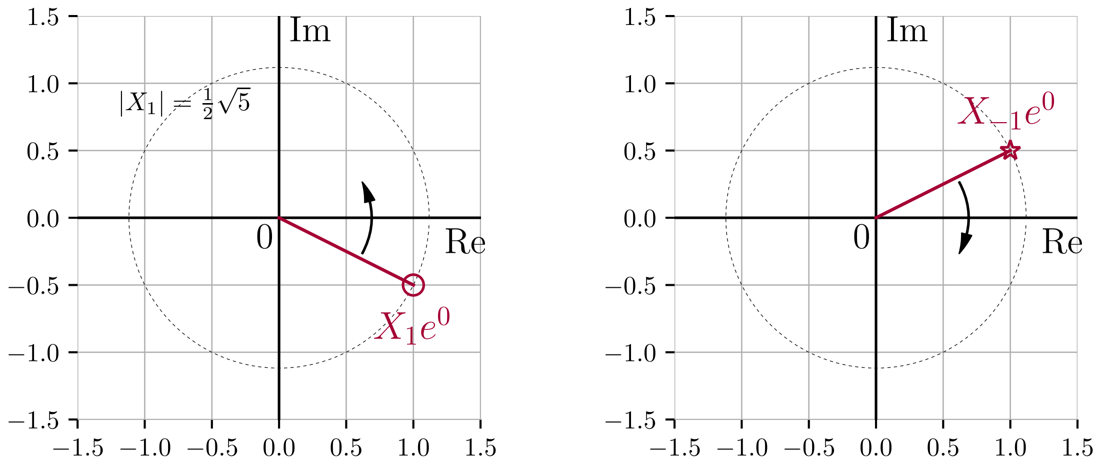
(a) The cosine and sine components of $x(t)$ have the same frequency, 2 Hz, so $f_0 = 2$ Hz. The real Fourier series coefficients can be directly obtained:

$a_1 = 2$ and $b_1 = 1$ Then: $X_1 = \frac{1}{2}(2-j)$ and $X_{-1} = \frac{1}{2}(2+j)$ (4.4)

Signal $x(t)$ can be expressed as the complex exponential Fourier series as follows:

$$x(t) = X_1 e^{j2\pi(2)t} + X_{-1}e^{-j2\pi(2)t} = \tfrac{1}{2}(2-j)e^{j4\pi t} + \tfrac{1}{2}(2+j)e^{-j4\pi t}$$

(b) The two complex conjugates are shown in the figure below, at left the counterclockwise-rotating vector $X_1 e^{j4\pi t}$ (the burgundy circle), at right the clockwise-rotating vector $X_{-1}e^{-j4\pi t}$ (the star), both at $t = 0$ s. Both vectors rotate in the complex plane along the circles indicated by the black dashed lines, with radius $|X_1| = |X_{-1}| = \frac{1}{2}\sqrt{5}$, (4.7).

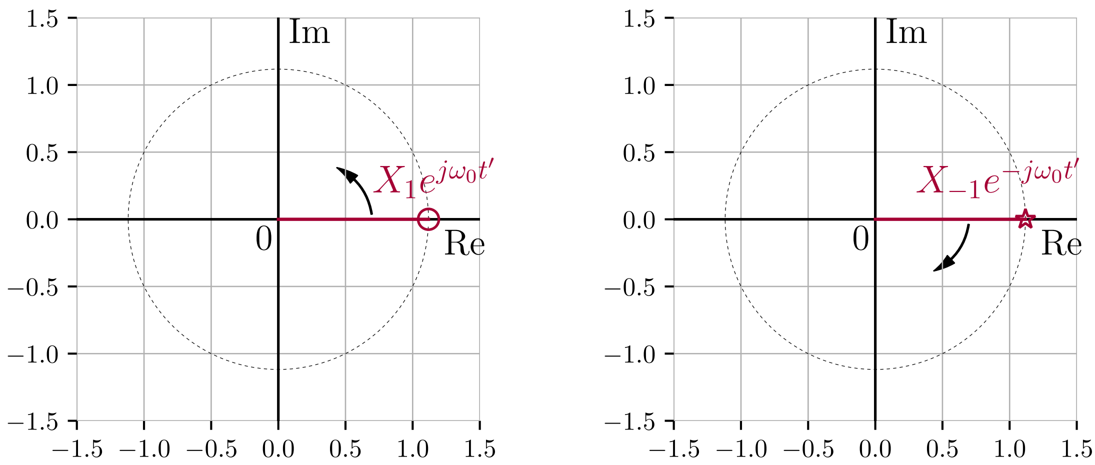


(c) Consider complex vector $X_1 e^{j4\pi t}$, which rotates counterclockwise in the complex plane. At $t = 0$ s, this complex vector has angle $\theta_1 = \arctan\left(\frac{\mathrm{Im}(X_1)}{\mathrm{Re}(X_1)}\right) = \arctan\left(-\frac{1}{2}\right)$, (4.8). With a rotation rate of 4π rad/s the vector will be aligned with the real axis when $4\pi t' = \arctan\left(\frac{1}{2}\right)$, hence at:

$$t' = \frac{1}{4\pi}\arctan\left(\frac{1}{2}\right) = 0.0369 \text{ s}$$

The complex conjugate, $X_{-1}e^{-j4\pi t}$, which rotates clockwise in the complex plane, will be aligned with the real axis at the same time.

The positions of the two complex vectors at $t' = 0.0369$ s are shown in the figure below, together with their rotation directions.



(d) The time t' equals the time shift $-t_0$ found in Solution $\mathcal{S}$.3-10(c). Clearly, a time shift t_0 in a harmonic causes the complex conjugate phasors describing that harmonic, (2.10), to rotate away (in opposite directions) from the real axis. Given that the complex exponential Fourier series coefficients are the same as the complex conjugate phasor pair, see Solution $\mathcal{S}$.4-2, this result should not be surprising. The coefficient $X_1 = \frac{1}{2}\vec{X}_1$ scales the magnitude of the complex exponential basis function $e^{j2\pi f_0 t}$ (a unit vector in the complex plane) from 1 to $|X_1|$ and sets its phase angle at $t = 0$ s from 0 to $\theta_1 = \angle X_1$, (4.6). The coefficient $X_{-1} = \frac{1}{2}\vec{X}_{-1}$ scales the magnitude of the complex exponential basis function $e^{-j2\pi f_0 t}$ from 1 to $|X_{-1}| = |X_1|$ and sets its phase angle at $t = 0$ s from 0 to $\theta_{-1} = \angle X_{-1} = -\theta_1$.

Consider the figure on page 54, where signal $x(t)$ is shown in burgundy. A time shift $t_0 = -0.0369$ s means that $x(t)$, which can be described as a cosine, is shifted to the right of $t = 0$ s, a negative phase shift. It reaches its maximum $|t_0|$ s 'later' than a zero-phase cosine (blue dashed curve). In the complex plane, at $t = 0$ s the complex conjugates have been rotated away from the real axis, in such a way that at $t = t'$ $(= -t_0)$ they cross the positive real axis and their sum reaches the maximum positive value.

$\mathcal{S}$.4-4

(a) The fundamental frequency of the complex exponential Fourier series equals $\omega_0 = \frac{5}{2}\pi$ rad/s. Hence: $f_0 = \frac{5}{4}$ Hz and period $T_0 = \frac{4}{5} = 0.8$ s

(b) The synthesis equation (4.2) states: $x(t) = \sum_{k=-\infty}^{\infty} X_k e^{jk\omega_0 t}$. Hence:

$$X_k = \frac{1}{1+j\pi k} \qquad \text{Fill in } k=0\text{:} \qquad X_0 = \frac{1}{1+j\pi 0} = 1$$

As X_0 represents the signal average, the average value of $x(t)$ over one period equals 1.

(c) The fifth harmonic ($k = \pm 5$) equals:

$$X_{-5}e^{-j5\frac{5}{2}\pi t} + X_5 e^{+j5\frac{5}{2}\pi t} = |X_{-5}|e^{j\angle X_{-5}}e^{-j\frac{25}{2}\pi t} + |X_5|e^{j\angle X_5}e^{+j\frac{25}{2}\pi t}$$
$$= |X_5|e^{-j(\frac{25}{2}\pi t + \angle X_5)} + |X_5|e^{j(\frac{25}{2}\pi t+\angle X_5)}$$

where we used the fact that for real-valued signals, $|X_{-5}| = |X_5|$ and $\angle X_{-5} = -\angle X_5$.

The harmonic can also be written as a cosine: $\overbrace{2|X_5|}^{\text{amplitude}} \cos\left(\frac{25}{2}\pi t + \overbrace{\angle X_5}^{\text{phase}}\right)$ (*)

The magnitude and phase of the fifth harmonic component can be computed using the real and imaginary parts:

$$X_5 = \frac{1}{1+j5\pi} = \frac{1}{1+j5\pi}\frac{1-j5\pi}{1-j5\pi} = \frac{1-j5\pi}{1+25\pi^2}$$

$$\mathrm{Re}(X_5) = \frac{1}{1+25\pi^2} \qquad \text{and} \qquad \mathrm{Im}(X_5) = \frac{-5\pi}{1+25\pi^2}$$

Now, using (4.7) and (4.8), respectively, we can compute the magnitude and phase:

$$|X_5| = |X_{-5}| = \sqrt{\mathrm{Re}^2(X_5) + \mathrm{Im}^2(X_5)} = \frac{1}{\sqrt{1+25\pi^2}}$$

$$\angle X_5 = -\angle X_{-5} = \arctan\left(\frac{\mathrm{Im}(X_5)}{\mathrm{Re}(X_5)}\right) = -\arctan(5\pi)$$

Hence, using (*), the amplitude of the fifth harmonic component equals $\frac{2}{\sqrt{1+25\pi^2}}$, and its phase equals $-\arctan(5\pi)$.

(d) Using the results obtained in (c):

$$x_{k=\pm5}(t) = X_{-5}e^{-j5\frac{5}{2}\pi t} + X_5 e^{+j5\frac{5}{2}\pi t}$$
$$= \frac{1}{\sqrt{1+25\pi^2}}e^{j\arctan(5\pi)}e^{-j5\frac{5}{2}\pi t} + \frac{1}{\sqrt{1+25\pi^2}}e^{-j\arctan(5\pi)}e^{j5\frac{5}{2}\pi t}$$
$$= \frac{1}{\sqrt{1+25\pi^2}}\underbrace{\left(e^{-j(5\frac{5}{2}\pi t - \arctan(5\pi))} + e^{j(5\frac{5}{2}\pi t - \arctan(5\pi))}\right)}_{2\cos\left(5\frac{5}{2}\pi t - \arctan(5\pi)\right)}$$
$$= \frac{2}{\sqrt{1+25\pi^2}}\cos\left(\frac{25}{2}\pi t - \arctan(5\pi)\right)$$

This expression can also be directly obtained by direct substitution of the magnitude and phase obtained in (c) into the cosine expression (*) also mentioned in (c).

s|4.3 Effects of symmetry

S.4-5

(a) Remember: an *even* function is a function where $x(-t) = x(t)$. An example is the cosine function.
Use Euler's formula $e^{j\theta} = \cos\theta + j\sin\theta$, (C.1), to rewrite (4.3):

$$X_k = \frac{1}{T_0}\int_{T_0} x(t)e^{-jk\omega_0 t}\,dt = \frac{1}{T_0}\int_{T_0} x(t)\big(\cos(k\omega_0 t) - j\sin(k\omega_0 t)\big)\,dt$$

$$\mathrm{Re}(X_k) = \frac{1}{T_0}\int_{T_0} x(t)\cos(k\omega_0 t)\,dt \qquad \text{and} \qquad \mathrm{Im}(X_k) = \frac{-1}{T_0}\int_{T_0} x(t)\sin(k\omega_0 t)\,dt$$

X_k is real when its imaginary part is zero, so when:

$$\frac{-1}{T_0}\int_{T_0} x(t)\sin(k\omega_0 t)\,dt = 0 \qquad \text{or, more specifically,} \qquad \int_{T_0} x(t)\sin(k\omega_0 t)\,dt = 0$$

This relation should hold for an integration over an arbitrary period T_0. For a function to be even, its behavior for positive and negative time t is important, i.e., time relative to $t = 0$, so we take the integral to be symmetric about $t = 0$:

$$\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\sin(k\omega_0 t)\;\mathrm{d}t = \int_{-\frac{T_0}{2}}^{0} x(t)\sin(k\omega_0 t)\;\mathrm{d}t + \int_{0}^{\frac{T_0}{2}} x(t)\sin(k\omega_0 t)\;\mathrm{d}t = 0$$

In the first integral on the right-hand side, we substitute $t = -\sigma$, then $\mathrm{d}t = \mathrm{d}(-\sigma)$; then for $t = -\frac{T_0}{2}$, we get $\sigma = \frac{T_0}{2}$, and for $t = 0$, we get $\sigma = 0$:

$$\int_{\sigma=\frac{T_0}{2}}^{0} x(-\sigma)\sin(-k\omega_0\sigma)\;\mathrm{d}(-\sigma) + \int_{t=0}^{\frac{T_0}{2}} x(t)\sin(k\omega_0 t)\;\mathrm{d}t = 0$$

Now, look at the integrand in the first integral: (1) $\sin(-k\omega_0\sigma) = -\sin(k\omega_0\sigma)$ (a sine is an odd function), (2) reversing the interval limits causes an additional minus sign, and (3) $\mathrm{d}(-\sigma) = -\mathrm{d}\sigma$, so we get three minus signs in the first integral. This results in:

$$-\int_{\sigma=0}^{\frac{T_0}{2}} x(-\sigma)\sin(k\omega_0\sigma)\;\mathrm{d}\sigma + \int_{t=0}^{\frac{T_0}{2}} x(t)\sin(k\omega_0 t)\;\mathrm{d}t = 0$$

Then, we simply substitute $\sigma = t$ in the first integral,[1] then $\mathrm{d}\sigma = \mathrm{d}t$; for $\sigma = 0$, we get $t = 0$, and for $\sigma = \frac{T_0}{2}$, we get $t = \frac{T_0}{2}$. We obtain:

$$\int_{t=0}^{\frac{T_0}{2}} x(-t)\sin(k\omega_0 t)\;\mathrm{d}t = \int_{t=0}^{\frac{T_0}{2}} x(t)\sin(k\omega_0 t)\;\mathrm{d}t$$

This equality holds when $x(-t) = x(t)$, or in other words, when $x(t)$ is even. This proves that when $x(t)$ is even, the imaginary part of the complex exponential Fourier series will be zero, and hence, X_k will be a real and even function of k.

(b) Remember: an *odd* function is a function where $x(-t) = -x(t)$. An example is the sine function.
The proof for X_k being imaginary and odd if $x(t)$ is real and odd, is very similar to the proof given in (a) for X_k being real and even if $x(t)$ is real and even. X_k is imaginary when its real part equals zero:

$$\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\cos(k\omega_0 t)\;\mathrm{d}t = \int_{-\frac{T_0}{2}}^{0} x(t)\cos(k\omega_0 t)\;\mathrm{d}t + \int_{0}^{\frac{T_0}{2}} x(t)\cos(k\omega_0 t)\;\mathrm{d}t = 0$$

In the first integral on the right-hand side, we substitute $t = -\sigma$, then $\mathrm{d}t = \mathrm{d}(-\sigma)$; then for $t = -\frac{T_0}{2}$, we get $\sigma = \frac{T_0}{2}$, and for $t = 0$, we get $\sigma = 0$:

$$\int_{\sigma=\frac{T_0}{2}}^{0} x(-\sigma)\cos(-k\omega_0\sigma)\;\mathrm{d}(-\sigma) + \int_{t=0}^{\frac{T_0}{2}} x(t)\cos(k\omega_0 t)\;\mathrm{d}t = 0, \qquad \text{therefore:}$$

$$\int_{\sigma=0}^{\frac{T_0}{2}} x(-\sigma)\cos(k\omega_0\sigma)\;\mathrm{d}\sigma + \int_{t=0}^{\frac{T_0}{2}} x(t)\cos(k\omega_0 t)\;\mathrm{d}t = 0$$

Then, we simply substitute $\sigma = t$ in the first integral, and we obtain:

$$\int_{t=0}^{\frac{T_0}{2}} x(-t)\cos(k\omega_0 t)\;\mathrm{d}t = -\int_{t=0}^{\frac{T_0}{2}} x(t)\cos(k\omega_0 t)\;\mathrm{d}t$$

This equality holds when $x(-t) = -x(t)$, or in other words, when $x(t)$ is odd. This proves that when $x(t)$ is odd, the real part of the complex exponential Fourier series will be zero, and hence, X_k will be imaginary and odd.

(c) Remember: a *half-wave odd* function is a function where $x\left(t \pm \frac{T_0}{2}\right) = -x(t)$.
Use Euler's formula $e^{j\theta} = \cos\theta + j\sin\theta$, (C.1), again to rewrite (4.3):

$$X_k = \frac{1}{T_0}\int_{T_0} x(t)e^{-jk\omega_0 t}\,\mathrm{d}t = \frac{1}{T_0}\int_{T_0} x(t)\big(\cos(k\omega_0 t) - j\sin(k\omega_0 t)\big)\,\mathrm{d}t$$

Hence:

$$\begin{aligned} X_k =& \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{0} \big(x(t)\cos(k\omega_0 t) - jx(t)\sin(k\omega_0 t)\big)\,\mathrm{d}t \\ &+ \frac{1}{T_0}\int_{0}^{\frac{T_0}{2}} \big(x(t)\cos(k\omega_0 t) - jx(t)\sin(k\omega_0 t)\big)\,\mathrm{d}t \end{aligned}$$

Again we choose the integration interval to be symmetric about $t = 0$. But now we substitute in the first integral $t = \sigma - \frac{T_0}{2}$, then $\mathrm{d}t = \mathrm{d}\sigma$; for $t = -\frac{T_0}{2}$, we get $\sigma = 0$, and for $t = 0$, we get $\sigma = \frac{T_0}{2}$. Hence, we obtain:

$$\begin{aligned} X_k =& \frac{1}{T_0}\int_{\sigma=0}^{\frac{T_0}{2}} \Big(x\big(\sigma - \tfrac{T_0}{2}\big)\cos\Big(k\omega_0\big(\sigma - \tfrac{T_0}{2}\big)\Big) - jx\big(\sigma - \tfrac{T_0}{2}\big)\sin\Big(k\omega_0\big(\sigma - \tfrac{T_0}{2}\big)\Big)\Big)\,\mathrm{d}\sigma \\ &+ \frac{1}{T_0}\int_{t=0}^{\frac{T_0}{2}} \big(x(t)\cos(k\omega_0 t) - jx(t)\sin(k\omega_0 t)\big)\,\mathrm{d}t \end{aligned}$$

Now we simply substitute $\sigma = t$ in the first integral, and we obtain:

$$\begin{aligned} X_k =& \frac{1}{T_0}\int_{t=0}^{\frac{T_0}{2}} \Big(x\big(t - \tfrac{T_0}{2}\big)\cos\Big(k\omega_0\big(t - \tfrac{T_0}{2}\big)\Big) - jx\big(t - \tfrac{T_0}{2}\big)\sin\Big(k\omega_0\big(t - \tfrac{T_0}{2}\big)\Big)\Big)\,\mathrm{d}t \\ &+ \frac{1}{T_0}\int_{t=0}^{\frac{T_0}{2}} \big(x(t)\cos(k\omega_0 t) - jx(t)\sin(k\omega_0 t)\big)\,\mathrm{d}t \end{aligned}$$

Use the half-wave odd symmetry property: $x\big(t + \frac{T_0}{2}\big) = -x(t)$. Because the signal is periodic with T_0, $x\big(t - \frac{T_0}{2}\big) = -x(t)$ also holds. We then rearrange:

$$\begin{aligned} X_k = \frac{1}{T_0}\int_{t=0}^{\frac{T_0}{2}} \Big(x(t)&\Big(\cos(k\omega_0 t) - \cos\Big(k\omega_0\big(t - \tfrac{T_0}{2}\big)\Big)\Big) \\ &- jx(t)\Big(\sin(k\omega_0 t) - \sin\Big(k\omega_0\big(t - \tfrac{T_0}{2}\big)\Big)\Big)\Big)\,\mathrm{d}t \end{aligned} \qquad (*)$$

Using trigonometric identity $\cos(u - v) = \cos u\cos v + \sin u\sin v$, (A.7), and $\omega_0 = \frac{2\pi}{T_0}$:

$$\begin{aligned} \cos\Big(k\omega_0\big(t - \tfrac{T_0}{2}\big)\Big) &= \cos\big(k\omega_0 t - k\omega_0\tfrac{T_0}{2}\big) \\ &= \cos(k\omega_0 t)\cos\big(k\omega_0\tfrac{T_0}{2}\big) + \sin(k\omega_0 t)\sin\big(k\omega_0\tfrac{T_0}{2}\big) \\ &= \cos\big(k\tfrac{2\pi}{T_0}t\big)\cos\big(k\tfrac{2\pi}{T_0}\tfrac{T_0}{2}\big) + \sin\big(k\tfrac{2\pi}{T_0}t\big)\sin\big(k\tfrac{2\pi}{T_0}\tfrac{T_0}{2}\big) \\ &= \cos\big(k\tfrac{2\pi}{T_0}t\big)\cos(k\pi) + \sin\big(k\tfrac{2\pi}{T_0}t\big)\sin(k\pi) \\ &= \cos(k\omega_0 t)\cos(k\pi) \qquad \text{as } \sin(k\pi) \text{ is } 0 \text{ for all values of } k. \end{aligned}$$

For even k, we obtain: $\cos\Big(k\omega_0\big(t - \frac{T_0}{2}\big)\Big) = \cos(k\omega_0 t)$

For odd k, we obtain: $\cos\Big(k\omega_0\big(t - \frac{T_0}{2}\big)\Big) = -\cos(k\omega_0 t)$

Similarly, using trigonometric identity $\sin(u - v) = \sin u\cos v - \cos u\sin v$, (A.6):

$$\begin{aligned} \sin\Big(k\omega_0\big(t - \tfrac{T_0}{2}\big)\Big) &= \sin\big(k\omega_0 t - k\omega_0\tfrac{T_0}{2}\big) \\ &= \sin(k\omega_0 t)\cos\big(k\omega_0\tfrac{T_0}{2}\big) - \cos(k\omega_0 t)\sin\big(k\omega_0\tfrac{T_0}{2}\big) \\ &= \sin\big(k\tfrac{2\pi}{T_0}t\big)\cos\big(k\tfrac{2\pi}{T_0}\tfrac{T_0}{2}\big) - \cos\big(k\tfrac{2\pi}{T_0}t\big)\sin\big(k\tfrac{2\pi}{T_0}\tfrac{T_0}{2}\big) \\ &= \sin\big(k\tfrac{2\pi}{T_0}t\big)\cos(k\pi) - \cos\big(k\tfrac{2\pi}{T_0}t\big)\sin(k\pi) \\ &= \sin(k\omega_0 t)\cos(k\pi) \qquad \text{as } \sin(k\pi) \text{ is } 0 \text{ for all values of } k. \end{aligned}$$

For even k, we obtain: $\sin\left(k\omega_0\left(t-\frac{T_0}{2}\right)\right)=\sin(k\omega_0 t)$

For odd k, we obtain: $\sin\left(k\omega_0\left(t-\frac{T_0}{2}\right)\right)=-\sin(k\omega_0 t)$

Substituting this into our main result so far (*), we see that:
For *even* k:

$$X_k=\frac{1}{T_0}\int_{t=0}^{\frac{T_0}{2}}\Big(x(t)\big(\cos(k\omega_0 t)-\cos(k\omega_0 t)\big)-jx(t)\big(\sin(k\omega_0 t)-\sin(k\omega_0 t)\big)\Big)\,\mathrm{d}t=0 \textbf{ qed}$$

For *odd* k:

$$X_k=\frac{1}{T_0}\int_{t=0}^{\frac{T_0}{2}}\Big(x(t)\big(\cos(k\omega_0 t)+\cos(k\omega_0 t)\big)-jx(t)\big(\sin(k\omega_0 t)+\sin(k\omega_0 t)\big)\Big)\,\mathrm{d}t$$

$$=\frac{2}{T_0}\int_{t=0}^{\frac{T_0}{2}}\big(x(t)\cos(k\omega_0 t)-jx(t)\sin(k\omega_0 t)\big)\,\mathrm{d}t$$

This proves that when $x(t)$ has half-wave odd symmetry, $X_k=0$ for even k.

Note that half-wave odd symmetry says nothing about the fact whether a signal is even or odd.[2]

[1]If you do not understand this substitution, please have a look at the footnote at the end of Solution S.3-6.

[2]From the proof in (c) we can also find another way to state the proofs in (a) and (b). We start with:

$$X_k=\frac{1}{T_0}\int_{t=-\frac{T_0}{2}}^{0}\big(x(t)\cos(k\omega_0 t)-jx(t)\sin(k\omega_0 t)\big)\,\mathrm{d}t+\frac{1}{T_0}\int_{t=0}^{\frac{T_0}{2}}\big(x(t)\cos(k\omega_0 t)-jx(t)\sin(k\omega_0 t)\big)\,\mathrm{d}t$$

In the first integral, we substitute $t=-\sigma$, then $\mathrm{d}t=\mathrm{d}(-\sigma)=-\mathrm{d}\sigma$; then for $t=-\frac{T_0}{2}$, we get $\sigma=\frac{T_0}{2}$, and for $t=0$, we get $\sigma=0$:

$$X_k=\frac{1}{T_0}\int_{\sigma=\frac{T_0}{2}}^{0}\big(x(-\sigma)\cos(-k\omega_0\sigma)-jx(-\sigma)\sin(-k\omega_0\sigma)\big)(-\mathrm{d}\sigma)$$
$$+\frac{1}{T_0}\int_{t=0}^{\frac{T_0}{2}}\big(x(t)\cos(k\omega_0 t)-jx(t)\sin(k\omega_0 t)\big)\,\mathrm{d}t$$

We rearrange:

$$X_k=\frac{1}{T_0}\int_{\sigma=0}^{\frac{T_0}{2}}\big(x(-\sigma)\cos(k\omega_0\sigma)+jx(-\sigma)\sin(k\omega_0\sigma)\big)\,\mathrm{d}\sigma+\frac{1}{T_0}\int_{t=0}^{\frac{T_0}{2}}\big(x(t)\cos(k\omega_0 t)-jx(t)\sin(k\omega_0 t)\big)\,\mathrm{d}t$$

And then we substitute $\sigma=t$ in the first integral:

$$X_k=\frac{1}{T_0}\int_{t=0}^{\frac{T_0}{2}}\big(x(-t)\cos(k\omega_0 t)+jx(-t)\sin(k\omega_0 t)\big)\,\mathrm{d}t+\frac{1}{T_0}\int_{t=0}^{\frac{T_0}{2}}\big(x(t)\cos(k\omega_0 t)-jx(t)\sin(k\omega_0 t)\big)\,\mathrm{d}t$$

Rearranging gives:

$$X_k=\frac{1}{T_0}\int_{t=0}^{\frac{T_0}{2}}\big(x(t)+x(-t)\big)\cos(k\omega_0 t)\,\mathrm{d}t-j\frac{1}{T_0}\int_{t=0}^{\frac{T_0}{2}}\big(x(t)-x(-t)\big)\sin(k\omega_0 t)\,\mathrm{d}t$$

From which we can see that when $x(t)$ is even, the term in the second integral becomes zero, and when $x(t)$ is odd, the term in the first integral becomes zero. **qed**

S.4-6

(a) The coefficient a_0 represents the mean of a signal. In this case, since the mean of $x(t)$ equals zero, $a_0=0$.

(b) The a_k coefficients belong to the cosine terms in the trigonometric Fourier series. Since $x(t)$ is real and odd-symmetric about $t=0$, we do not need the cosine terms to construct the signal model, and therefore all a_k's are zero.

(c) Since $x(t)$ is real and odd-symmetric about $t=0$, we only need the sine terms in the trigonometric Fourier series to model this signal, i.e., the b_k's. From (4.4), we know the relationship between the complex exponential Fourier series coefficients X_k and the trigonometric Fourier series coefficients:

$X_0 = a_0$ and $X_k = \frac{1}{2}(a_k - jb_k)$ and $X_{-k} = \frac{1}{2}(a_k + jb_k)$ for $k \in \mathbb{N}^+$

Clearly, since all a_k's are zero:

$$X_k = -j\frac{b_k}{2} \quad \text{and} \quad X_{-k} = j\frac{b_k}{2} \quad \text{for} \quad k \in \mathbb{N}^+$$

Therefore, the complex exponential Fourier series coefficients X_k are imaginary. This is simply a consequence of the fact that $x(t)$ is odd.

(d) From the illustration, we can see that $x(t)$ has half-wave odd symmetry, i.e., $x(t \pm \frac{T_0}{2}) = -x(t)$. As a consequence, all even-indexed coefficients X_k in the complex exponential Fourier series are zero.

S.4-7

(I) True, as X_0 represents the signal's average, which for $x(t)$ equals b.

(II) True, as all coefficients Y_k (except $k = 0$) of a signal $y(t)$ will be purely imaginary when that signal is odd. Despite, strictly mathematically, $x(t)$ not being odd, the only property 'preventing' $x(t)$ from being odd is the signal's average b, which only has an influence on X_0. All other coefficients still have the characteristics of an odd signal, hence, all coefficients X_k (except $k = 0$) will still be purely imaginary.

S.4-8

(1) For a signal to have purely real coefficients, the signal should be real and even. All signals in this book are real, however, only signals $c(t)$ and $f(t)$ are even. Therefore, only $c(t)$ and $f(t)$ have purely real coefficients.

(2) For a signal to have purely imaginary coefficients, the signal should be real and odd. Only signal $b(t)$ is odd, hence, that is the only signal with purely imaginary coefficients.

(3) If a signal is neither odd, nor even, its Fourier series coefficients are complex. Signals $a(t)$, $d(t)$ and $e(t)$ are neither odd, nor even, and thus have complex coefficients.

(4) Even-indexed coefficients are zero when a signal has half-wave odd symmetry, i.e., $x(t \pm \frac{T_0}{2}) = -x(t)$. Signals $a(t)$, $b(t)$, $c(t)$ and $e(t)$ are half-wave odd symmetric, and therefore, $X_k = 0$ for even k. This is not the case for $d(t)$ and $f(t)$.

(5) Signals $a(t)$, $b(t)$, $c(t)$ and $e(t)$ have a mean of zero, so for them $X_0 = 0$. Signals $d(t)$ and $f(t)$ have a mean > 0, so for them $X_0 > 0$.

	Signal					
Property	(a)	(b)	(c)	(d)	(e)	(f)
(1) Purely real coefficients			X			X
(2) Purely imaginary coefficients		X				
(3) Complex coefficients	X			X	X	
(4) Even-indexed coefficients are zero	X	X	X		X	
(5) $X_0 = 0$	X	X	X		X	

S|4.5 Examples

S.4-9 We will do (b) and (c) in a different way than (a), and (d) in a different way than (a), (b) and (c) to show that you can apply different methods here.

(a) First use trigonometric identity (A.7):

$$\begin{aligned} x_a(t) &= 1 - 2\sin(5\pi t) + 3\cos\left(10\pi t + \tfrac{\pi}{3}\right) \\ &= 1 - 2\sin(5\pi t) + 3\cos(10\pi t)\cos\left(\tfrac{\pi}{3}\right) - 3\sin(10\pi t)\sin\left(\tfrac{\pi}{3}\right) \\ &= 1 - 2\sin(5\pi t) + \tfrac{3}{2}\cos(10\pi t) - \tfrac{3}{2}\sqrt{3}\sin(10\pi t) \end{aligned}$$

We recognize frequencies of 2.5 and 5 Hz, so the highest common frequency is 2.5 Hz. Then, using the fundamental frequency $\omega_0 = 2\pi \cdot 2.5 = 5\pi$ rad/s:

$$x_a(t) = 1 - 2\sin(\omega_0 t) + \tfrac{3}{2}\cos(2\omega_0 t) - \tfrac{3}{2}\sqrt{3}\sin(2\omega_0 t)$$

We obtain: $a_0 = 1$, $b_1 = -2$, $a_2 = \frac{3}{2}$ and $b_2 = -\frac{3}{2}\sqrt{3}$

Hence, using (4.4) we get for the complex exponential Fourier series coefficients:

$$X_0 = a_0 = 1$$

$$X_1 = \tfrac{1}{2}(a_1 - jb_1) = \tfrac{1}{2}\big(0 - j(-2)\big) = j \qquad \text{and} \qquad X_{-1} = X_1^* = -j$$

$$X_2 = \tfrac{1}{2}(a_2 - jb_2) = \tfrac{1}{2}\left(\tfrac{3}{2} - j(-\tfrac{3}{2}\sqrt{3}\,)\right) = \tfrac{3}{4} + j\tfrac{3}{4}\sqrt{3} \qquad \text{and} \qquad X_{-2} = X_2^* = \tfrac{3}{4} - j\tfrac{3}{4}\sqrt{3}$$

Hence, the complex exponential Fourier series coefficients are:

k	-2	-1	0	1	2
X_k	$\frac{3}{4} - j\frac{3}{4}\sqrt{3}$	$-j$	1	j	$\frac{3}{4} + j\frac{3}{4}\sqrt{3}$

(b) First use trigonometric identity (A.4) and then (A.10):

$$\begin{aligned} x_b(t) &= 5\cos^2(30\pi t)\sin(15\pi t) = \tfrac{5}{2}\sin(15\pi t) + \tfrac{5}{2}\sin(15\pi t)\cos(60\pi t) \\ &= \tfrac{5}{2}\sin(15\pi) + \tfrac{5}{4}\sin(-45\pi t) + \tfrac{5}{4}\sin(75\pi t) \\ &= \tfrac{5}{2}\sin(15\pi t) - \tfrac{5}{4}\sin(45\pi t) + \tfrac{5}{4}\sin(75\pi t) \end{aligned}$$

We recognize frequencies of 7.5, 22.5 and 37.5 Hz, so the highest common frequency is 7.5 Hz. Then, using the fundamental frequency $\omega_0 = 2\pi \cdot 7.5 = 15\pi$ rad/s:

$$x_b(t) = \tfrac{5}{2}\sin(\omega_0 t) - \tfrac{5}{4}\sin(3\omega_0 t) + \tfrac{5}{4}\sin(5\omega_0 t)$$

Using Euler's formula (C.3) and rearranging from negative to positive frequencies:

$$x_b(t) = \tfrac{5}{8}je^{-j5\omega_0 t} - \tfrac{5}{8}je^{-j3\omega_0 t} + \tfrac{5}{4}je^{-j1\omega_0 t} - \tfrac{5}{4}je^{j1\omega_0 t} + \tfrac{5}{8}je^{j3\omega_0 t} - \tfrac{5}{8}je^{j5\omega_0 t}$$

Hence, the complex exponential Fourier series coefficients are:

k	-5	-3	-1	1	3	5
X_k	$\frac{5}{8}j$	$-\frac{5}{8}j$	$\frac{5}{4}j$	$-\frac{5}{4}j$	$\frac{5}{8}j$	$-\frac{5}{8}j$

(c) First use trigonometric identity (A.5) and then (A.10):

$$\begin{aligned} x_c(t) &= 3\sin^3(35\pi t) + 4\cos(20\pi t) = 3\sin(35\pi t)\,\tfrac{1}{2}\big(1 - \cos(70\pi t)\big) + 4\cos(20\pi t) \\ &= \tfrac{3}{2}\sin(35\pi t) - \tfrac{3}{2}\sin(35\pi t)\cos(70\pi t) + 4\cos(20\pi t) \\ &= \tfrac{3}{2}\sin(35\pi t) - \tfrac{3}{4}\sin(-35\pi t) - \tfrac{3}{4}\sin(105\pi t) + 4\cos(20\pi t) \\ &= \tfrac{9}{4}\sin(35\pi t) - \tfrac{3}{4}\sin(105\pi t) + 4\cos(20\pi t) \end{aligned}$$

We recognize frequencies of 17.5, 52.5 and 10 Hz, so the highest common frequency is 2.5 Hz. Then, using the fundamental frequency $\omega_0 = 2\pi \cdot 2.5 = 5\pi$ rad/s:

$$x_c(t) = \frac{9}{4}\sin(7\omega_0 t) - \frac{3}{4}\sin(21\omega_0 t) + 4\cos(4\omega_0 t)$$

Using Euler's formula, (C.2) and (C.3), and rearranging from negative to positive frequencies:

$$x_c(t) = -\frac{3}{8}je^{-j21\omega_0 t} + \frac{9}{8}je^{-j7\omega_0 t} + 2e^{-j4\omega_0 t} + 2e^{j4\omega_0 t} - \frac{9}{8}je^{j7\omega_0 t} + \frac{3}{8}je^{j21\omega_0 t}$$

Hence, the complex exponential Fourier series coefficients are:

k	-21	-7	-4	4	7	21
X_k	$-\frac{3}{8}j$	$\frac{9}{8}j$	2	2	$-\frac{9}{8}j$	$\frac{3}{8}j$

(d) Let us directly move to the complex exponential functions:

$$x_d(t) = 2\sin^2(70\pi t)\cos^2(30\pi t) + 7\sin(20\pi t)\cos(5\pi t) = a(t) + b(t)$$

$$\begin{aligned}
a(t) &= 2\left(\frac{1}{2j}\right)^2\left(e^{j70\pi t} - e^{-j70\pi t}\right)^2\left(\frac{1}{2}\right)^2\left(e^{j30\pi t} + e^{-j30\pi t}\right)^2 \\
&= 2\left(\frac{1}{2j}\right)^2\left(e^{j140\pi t} - 2 + e^{-j140\pi t}\right)\left(\frac{1}{2}\right)^2\left(e^{j60\pi t} + 2 + e^{-j60\pi t}\right) \\
&= -\frac{1}{8}\left(e^{j200\pi t} + 2e^{j140\pi t} + e^{j80\pi t} - 2e^{j60\pi t} - 4\right. \\
&\qquad \left. - 2e^{-j60\pi t} + e^{-j80\pi t} + 2e^{-j140\pi t} + e^{-j200\pi t}\right) \\
&= \frac{1}{2} - \frac{1}{8}\left(e^{-j200\pi t} + e^{j200\pi t}\right) - \frac{1}{4}\left(e^{-j140\pi t} + e^{j140\pi t}\right) \\
&\qquad - \frac{1}{8}\left(e^{-j80\pi t} + e^{j80\pi t}\right) + \frac{1}{4}\left(e^{-j60\pi t} + e^{j60\pi t}\right) \\
b(t) &= 7\frac{1}{2j}\left(e^{j20\pi t} - e^{-j20\pi t}\right)\frac{1}{2}\left(e^{j5\pi t} + e^{-j5\pi t}\right) \\
&= -\frac{7}{4}j\left(-e^{-j25\pi t} - e^{-j15\pi t} + e^{j15\pi t} + e^{j25\pi t}\right)
\end{aligned}$$

Adding them together and rearranging from negative to positive frequencies:

$$\begin{aligned}
x_d(t) = &- \frac{1}{8}e^{-j200\pi t} - \frac{1}{4}e^{-j140\pi t} - \frac{1}{8}e^{-j80\pi t} + \frac{1}{4}e^{-j60\pi t} + \frac{7}{4}je^{-j25\pi t} + \frac{7}{4}je^{-j15\pi t} \\
&+ \frac{1}{2} - \frac{7}{4}je^{j15\pi t} - \frac{7}{4}je^{j25\pi t} + \frac{1}{4}e^{j60\pi t} - \frac{1}{8}e^{j80\pi t} - \frac{1}{4}e^{j140\pi t} - \frac{1}{8}e^{j200\pi t}
\end{aligned}$$

We recognize frequencies of 7.5, 12.5, 30, 40, 70 and 100 Hz, so the highest common frequency is 2.5 Hz. Then, using the fundamental frequency $\omega_0 = 2\pi \cdot 2.5 = 5\pi$ rad/s:

$$\begin{aligned}
x_d(t) = &- \frac{1}{8}e^{-j40\omega_0 t} - \frac{1}{4}e^{-j28\omega_0 t} - \frac{1}{8}e^{-j16\omega_0 t} + \frac{1}{4}e^{-j12\omega_0 t} + \frac{7}{4}je^{-j5\omega_0 t} + \frac{7}{4}je^{-j3\omega_0 t} \\
&+ \frac{1}{2} - \frac{7}{4}je^{j3\omega_0 t} - \frac{7}{4}je^{j5\omega_0 t} + \frac{1}{4}e^{j12\omega_0 t} - \frac{1}{8}e^{j16\omega_0 t} - \frac{1}{4}e^{j28\omega_0 t} - \frac{1}{8}e^{j40\omega_0 t}
\end{aligned}$$

Hence, the complex exponential Fourier series coefficients are:

k	-40	-28	-16	-12	-5	-3	0	3	5	12	16	28	40
X_k	$-\frac{1}{8}$	$-\frac{1}{4}$	$-\frac{1}{8}$	$\frac{1}{4}$	$\frac{7}{4}j$	$\frac{7}{4}j$	$\frac{1}{2}$	$-\frac{7}{4}j$	$-\frac{7}{4}j$	$\frac{1}{4}$	$-\frac{1}{8}$	$-\frac{1}{4}$	$-\frac{1}{8}$

S.4-10

(a) Using the analysis equation (4.3):

$$X_k = \frac{1}{T_0}\int_{T_0} x(t)e^{-jk\omega_0 t}\,\mathrm{d}t = \frac{1}{T_0}\int_0^{\frac{T_0}{2}} A\sin(\omega_0 t)\,e^{-jk\omega_0 t}\,\mathrm{d}t$$

Substituting $\sin(\omega_0 t) = j\frac{1}{2}\left(e^{-j\omega_0 t} - e^{j\omega_0 t}\right)$, (C.3), into the above equation:

$$X_k = j\frac{A}{2T_0}\int_0^{\frac{T_0}{2}} \left(e^{-j\omega_0 t} - e^{j\omega_0 t}\right) e^{-jk\omega_0 t}\,\mathrm{d}t = j\frac{A}{2T_0}\left(\int_0^{\frac{T_0}{2}} e^{-j\omega_0(1+k)t}\,\mathrm{d}t - \int_0^{\frac{T_0}{2}} e^{j\omega_0(1-k)t}\,\mathrm{d}t\right)$$

$$= j\frac{A}{2T_0}\left(\left[\frac{1}{-j\omega_0(1+k)}e^{-j\omega_0(1+k)t}\right]_0^{\frac{T_0}{2}} - \left[\frac{1}{j\omega_0(1-k)}e^{j\omega_0(1-k)t}\right]_0^{\frac{T_0}{2}}\right) \qquad \text{for } k \neq \pm 1$$

Substituting $\omega_0 = \frac{2\pi}{T_0}$ into the denominator and the exponents:

$$X_k = \frac{A}{4\pi}\left(\left[-\frac{e^{-j\frac{2\pi}{T_0}(1+k)t}}{1+k}\right]_0^{\frac{T_0}{2}} - \left[\frac{e^{j\frac{2\pi}{T_0}(1-k)t}}{1-k}\right]_0^{\frac{T_0}{2}}\right) = \frac{A}{4\pi}\left(-\frac{e^{-j\pi(1+k)} - 1}{1+k} - \frac{e^{j\pi(1-k)} - 1}{1-k}\right)$$

Both $e^{-j\pi(1+k)}$ and $e^{j\pi(1-k)}$ equal -1 for even indices k and 1 for odd indices k, and can thus be simplified to $-(-1)^k$:

$$X_k = \frac{A}{4\pi}\left(-\frac{-(-1)^k - 1}{1+k} - \frac{-(-1)^k - 1}{1-k}\right)$$

From this expression, it follows:

For k odd:

$$X_k = 0, \quad \text{where } k \neq \pm 1$$

For k even:

$$X_k = \frac{A}{4\pi}\left(-\frac{-1-1}{1+k} - \frac{-1-1}{1-k}\right) = \frac{A}{4\pi}\left(\frac{2(1-k) + 2(1+k)}{1-k^2}\right) = \frac{A}{\pi}\frac{1}{1-k^2}$$

The special cases of $k = 1$ and $k = -1$ must be handled separately:

$$X_1 = j\frac{A}{2T_0}\int_0^{\frac{T_0}{2}} \left(e^{-j\omega_0 t} - e^{j\omega_0 t}\right) e^{-j\omega_0 t}\,\mathrm{d}t = j\frac{A}{2T_0}\int_0^{\frac{T_0}{2}} \left(e^{-j2\omega_0 t} - 1\right)\mathrm{d}t$$

$$= j\frac{A}{2T_0}\left[\frac{1}{-j2\omega_0}e^{-j2\omega_0 t} - t\right]_0^{\frac{T_0}{2}} = j\frac{A}{2T_0}\left(-\frac{e^{-j\omega_0 T_0}}{j2\omega_0} - \frac{T_0}{2} + \frac{1}{j2\omega_0}\right)$$

Substituting $\omega_0 T_0 = 2\pi$ into the exponent and using $e^{-j2\pi} = 1$:

$$X_1 = j\frac{A}{2T_0}\left(-\frac{1}{j2\omega_0} - \frac{T_0}{2} + \frac{1}{j2\omega_0}\right) = -\frac{Aj}{4}$$

A similar integration and substitution for $k = -1$ yields:

$$X_{-1} = j\frac{A}{2T_0}\int_0^{\frac{T_0}{2}} \left(e^{-j\omega_0 t} - e^{j\omega_0 t}\right) e^{j\omega_0 t}\,\mathrm{d}t = j\frac{A}{2T_0}\int_0^{\frac{T_0}{2}} \left(1 - e^{j2\omega_0 t}\right)\mathrm{d}t$$

$$= j\frac{A}{2T_0}\left[t - \frac{1}{j2\omega_0}e^{j2\omega_0 t}\right]_0^{\frac{T_0}{2}} = j\frac{A}{2T_0}\left(\frac{T_0}{2} - \frac{e^{j\omega_0 T_0}}{j2\omega_0} + \frac{1}{j2\omega_0}\right) = \frac{Aj}{4}$$

Concluding:
$$X_k = \begin{cases} \dfrac{A}{\pi(1-k^2)} & \text{for } k \text{ even (including } k = 0) \\ 0 & \text{for } k \text{ odd and} \neq \pm 1 \\ -j\dfrac{Ak}{4} & \text{for } k = \pm 1 \end{cases}$$

(b) First note that the period of the full-rectified sine wave T_0 is not equal to the period of the non-rectified sine wave T_1. Hence, we first define the frequency of the non-rectified sine wave to be ω_1 and that of the full-rectified sine wave to be $\omega_0 = 2\omega_1$, since $T_0 = \frac{T_1}{2}$. Using the analysis equation (4.3):

$$X_k = \frac{1}{T_0}\int_{T_0} x(t)e^{-jk\omega_0 t}\,\mathrm{d}t = \frac{1}{T_0}\int_0^{T_0} A\sin(\omega_1 t)\, e^{-jk\omega_0 t}\,\mathrm{d}t$$

Substituting $\sin(\omega_1 t) = j\frac{1}{2}\left(e^{-j\omega_1 t} - e^{j\omega_1 t}\right)$, (C.3), into the above equation:

$$X_k = j\frac{A}{2T_0}\int_0^{T_0}\left(e^{-j\omega_1 t} - e^{j\omega_1 t}\right) e^{-jk\omega_0 t}\,\mathrm{d}t = j\frac{A}{2T_0}\left(\int_0^{T_0} e^{-j(\omega_1 + k\omega_0)t}\,\mathrm{d}t - \int_0^{T_0} e^{j(\omega_1 - k\omega_0)t}\,\mathrm{d}t\right)$$

Substituting $\omega_0 = 2\omega_1$ and $T_0 = \frac{T_1}{2} = \frac{2\pi}{2\omega_1} = \frac{\pi}{\omega_1}$:

$$X_k = j\frac{A\omega_1}{2\pi}\left(\int_0^{\frac{\pi}{\omega_1}} e^{-j\omega_1(1+2k)t}\,\mathrm{d}t - \int_0^{\frac{\pi}{\omega_1}} e^{j\omega_1(1-2k)t}\,\mathrm{d}t\right)$$

$$= j\frac{A\omega_1}{2\pi}\left(\left[\frac{1}{-j\omega_1(1+2k)}e^{-j\omega_1(1+2k)t}\right]_0^{\frac{\pi}{\omega_1}} - \left[\frac{1}{j\omega_1(1-2k)}e^{j\omega_1(1-2k)t}\right]_0^{\frac{\pi}{\omega_1}}\right)$$

$$= -\frac{A}{2\pi}\left(\frac{1}{1+2k}\left(e^{-j(1+2k)\pi} - 1\right) + \frac{1}{1-2k}\left(e^{j(1-2k)\pi} - 1\right)\right)$$

Let us have a look at the exponentials:

$$e^{-j(1+2k)\pi} = e^{-j\pi}e^{-jk2\pi} = (-1)e^{-jk2\pi} \quad \text{where} \quad e^{-jk2\pi} = 1 \;\; \forall \;\; k \in \mathbb{Z}$$

Hence, the exponential is $-1 \;\; \forall \;\; k \in \mathbb{Z}$. Similarly:

$$e^{j(1-2k)\pi} = e^{j\pi}e^{-jk2\pi} = (-1)e^{-jk2\pi} \quad \text{where} \quad e^{-jk2\pi} = 1 \;\; \forall \;\; k \in \mathbb{Z}$$

Again, the exponential is $-1 \;\; \forall \;\; k \in \mathbb{Z}$.

$$X_k = -\frac{A}{2\pi}\left(\frac{1}{1+2k}\big((-1) - 1\big) + \frac{1}{1-2k}\big((-1) - 1\big)\right) = \frac{A}{\pi}\left(\frac{1}{1+2k} + \frac{1}{1-2k}\right)$$

$$= \frac{A}{\pi}\left(\frac{1}{1+2k}\frac{1-2k}{1-2k} + \frac{1}{1-2k}\frac{1+2k}{1+2k}\right) = \frac{A}{\pi}\left(\frac{(1-2k)+(1+2k)}{1-4k^2}\right) = \frac{2A}{\pi(1-4k^2)}$$

Concluding: $X_k = \frac{2A}{\pi(1-4k^2)} \quad \forall \;\; k \in \mathbb{Z}$

As a quick sanity check: note that from the illustration of the waveform, we can already see that this (real) signal is even. Hence, its complex exponential Fourier series coefficients are all real and even. Indeed, the above-derived complex exponential Fourier series coefficients are all real (imaginary parts equal zero). They are also even, since k appears as k^2. Hence, for positive and negative k, we obtain the same values.

(c) Let us obtain the complex exponential Fourier series coefficients of the square wave using two different methods. One method would be to take the trigonometric Fourier series coefficients of this signal, which have been obtained in Exercise *e*.3-4(a), as a starting point and then put them together to obtain the complex exponential Fourier series coefficients X_k. The other method would be to use the analysis equation (4.3). In the following, both methods will be shown.

Method 1: Put trigonometric Fourier series coefficients together

Take as a starting point the trigonometric Fourier series coefficients of this signal, which have been obtained in Exercise *e*.3-4(a) (see Solution *s*.3-4(a)):

$$a_k = \begin{cases} \frac{4A}{k\pi} & \text{for } \; k = 1, 5, 9, \ldots \\ -\frac{4A}{k\pi} & \text{for } \; k = 3, 7, 11, \ldots \\ 0 & \text{for } \; k \text{ even (including } k = 0) \end{cases} \qquad \text{all } \; b_k = 0$$

Now, let us convert these coefficients into the complex exponential Fourier series coefficients using (4.4):

$$X_0 = a_0 \qquad \text{and} \qquad X_k = \tfrac{1}{2}(a_k - jb_k) \qquad \text{and} \qquad X_{-k} = \tfrac{1}{2}(a_k + jb_k) \qquad \text{for} \;\; k \in \mathbb{N}^+$$

We directly obtain the desired result by substitution of the a_k's and b_k's in the above:

$$X_k = \begin{cases} \frac{2A}{|k|\pi} & \text{for } \; k = \pm1, \pm5, \pm9, \ldots \\ -\frac{2A}{|k|\pi} & \text{for } \; k = \pm3, \pm7, \pm11, \ldots \\ 0 & \text{for } \; k \text{ even (including } k = 0) \end{cases}$$

As a quick sanity check: note that from the illustration of the waveform, we can already see that this (real) signal is even. Hence, its complex exponential Fourier series coefficients are all real and even. Furthermore, the signal also has half-wave odd symmetry. Hence, all even-indexed coefficients are zero. Indeed, the above-derived coefficients are zero for k even. Furthermore, for k odd, all coefficients are real (imaginary parts equal zero). They are also even, as for positive and negative k, we obtain the same values.

Method 2: Direct calculation of complex exponential Fourier series coefficients

For the second method, let us start 'from scratch'. Since the illustration shows that this square wave is even, its complex exponential Fourier series coefficients must be real. Also, as a consequence of the signal being even, we must obtain the same values for positive and negative k. Next to being even, this signal is also half-wave odd. As a result, all even-indexed coefficients must be zero. This includes $k = 0$, since the signal has a zero mean, so $X_0 = 0$.

Let us now continue with the analysis equation (4.3):

$$
\begin{aligned}
X_k &= \frac{1}{T_0}\int_{T_0} x(t)e^{-jk\omega_0 t}\,\mathrm{d}t \\
&= \frac{1}{T_0}\left(\int_{-\frac{T_0}{2}}^{-\frac{T_0}{4}} (-A)e^{-jk\omega_0 t}\,\mathrm{d}t + \int_{-\frac{T_0}{4}}^{\frac{T_0}{4}} Ae^{-jk\omega_0 t}\,\mathrm{d}t + \int_{\frac{T_0}{4}}^{\frac{T_0}{2}} (-A)e^{-jk\omega_0 t}\,\mathrm{d}t\right) \\
&= \frac{A}{T_0}\left(\left[-\frac{1}{-jk\omega_0}e^{-jk\omega_0 t}\right]_{-\frac{T_0}{2}}^{-\frac{T_0}{4}} + \left[\frac{1}{-jk\omega_0}e^{-jk\omega_0 t}\right]_{-\frac{T_0}{4}}^{\frac{T_0}{4}} + \left[-\frac{1}{-jk\omega_0}e^{-jk\omega_0 t}\right]_{\frac{T_0}{4}}^{\frac{T_0}{2}}\right)
\end{aligned}
$$

Using $\omega_0 = \frac{2\pi}{T_0}$:

$$
\begin{aligned}
X_k &= \frac{A}{T_0}\left(\left[\frac{1}{jk\frac{2\pi}{T_0}}e^{-jk\frac{2\pi}{T_0}t}\right]_{-\frac{T_0}{2}}^{-\frac{T_0}{4}} + \left[\frac{1}{-jk\frac{2\pi}{T_0}}e^{-jk\frac{2\pi}{T_0}t}\right]_{-\frac{T_0}{4}}^{\frac{T_0}{4}} + \left[\frac{1}{jk\frac{2\pi}{T_0}}e^{-jk\frac{2\pi}{T_0}t}\right]_{\frac{T_0}{4}}^{\frac{T_0}{2}}\right) \\
&= \frac{A}{jk2\pi}\left(e^{jk\frac{2\pi}{T_0}\frac{T_0}{4}} - e^{jk\frac{2\pi}{T_0}\frac{T_0}{2}} - e^{-jk\frac{2\pi}{T_0}\frac{T_0}{4}} + e^{jk\frac{2\pi}{T_0}\frac{T_0}{4}} + e^{-jk\frac{2\pi}{T_0}\frac{T_0}{2}} - e^{-jk\frac{2\pi}{T_0}\frac{T_0}{4}}\right) \\
&= \frac{A}{jk2\pi}\left(2e^{jk\frac{\pi}{2}} - e^{jk\pi} - 2e^{-jk\frac{\pi}{2}} + e^{-jk\pi}\right)
\end{aligned}
$$

Note that $e^{jk\pi}$ and $e^{-jk\pi}$ both result in -1 for odd indices k and 1 for even indices k. Therefore, the terms $-e^{jk\pi}$ and $+e^{-jk\pi}$ always cancel each other out.

Then: $\quad X_k = \frac{A}{jk\pi}\left(e^{jk\frac{\pi}{2}} - e^{-jk\frac{\pi}{2}}\right)$

Now make a table (see on the right) to see what the remaining terms become for different values of k. Note that all even-indexed coefficients remain to be zero as a consequence of the signal being even (when filling in even values for k, the two remaining terms would cancel each other out). Further note that only a small subset of k's is needed to 'get an idea of what is happening'. As can be deduced from the table:

$$
X_k = \begin{cases} \frac{2A}{|k|\pi} & \text{for } k = \pm 1, \pm 5, \pm 9, \dots \\ -\frac{2A}{|k|\pi} & \text{for } k = \pm 3, \pm 7, \pm 11, \dots \\ 0 & \text{for } k \text{ even (including } k = 0) \end{cases}
$$

k	$e^{jk\frac{\pi}{2}}$	$e^{-jk\frac{\pi}{2}}$	X_k
-7	j	$-j$	$-\frac{2A}{7\pi}$
-5	$-j$	j	$\frac{2A}{5\pi}$
-3	j	$-j$	$-\frac{2A}{3\pi}$
-1	$-j$	j	$\frac{2A}{1\pi}$
1	j	$-j$	$\frac{2A}{1\pi}$
3	$-j$	j	$-\frac{2A}{3\pi}$
5	j	$-j$	$\frac{2A}{5\pi}$
7	$-j$	j	$-\frac{2A}{7\pi}$

This is the same answer as that obtained using Method 1, as it should be. However, note that (as expected) 'starting from scratch' takes a lot more effort than converting trigonometric Fourier series coefficients into complex exponential coefficients, if one has already obtained the trigonometric coefficients.

(d) We can obtain the complex exponential Fourier series coefficients of the sawtooth waveform using two different methods. The most straightforward method would be to use the analysis equation (4.3). Another method would be to compute the trigonometric Fourier series coefficients a_k and b_k and then put them together to obtain the complex exponential Fourier series coefficients X_k. In the following, both methods will be shown.

Method 1: Direct calculation of complex exponential Fourier series coefficients

Using the analysis equation (4.3):

$$X_k = \frac{1}{T_0}\int_{T_0} x(t)e^{-jk\omega_0 t}\,\mathrm{d}t = \frac{A}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} te^{-jk\omega_0 t}\,\mathrm{d}t$$

We then use the partial integration rule that states that $\int_a^b \frac{\mathrm{d}f(x)}{\mathrm{d}x}g(x)\,\mathrm{d}x = [f(x)g(x)]_a^b - \int_a^b f(x)\frac{\mathrm{d}g(x)}{\mathrm{d}x}\,\mathrm{d}x$: define $\frac{\mathrm{d}f(t)}{\mathrm{d}t} = e^{-jk\omega_0 t}$ and $g(t) = t$, from which it follows that $f(t) = -\frac{1}{jk\omega_0}e^{-jk\omega_0 t}$ and $\frac{\mathrm{d}g(t)}{\mathrm{d}t} = 1$, and we obtain:

$$X_k = -\frac{A}{T_0 jk\omega_0}\left(\left[te^{-jk\omega_0 t}\right]_{-\frac{T_0}{2}}^{\frac{T_0}{2}} - \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} e^{-jk\omega_0 t}\,\mathrm{d}t\right)$$

The remaining integral on the right-hand side equals zero for all indices k, as this is simply the complex equivalent of integrating a cosine or sine function over one period, where the frequencies of the sines and cosines equal an integer multiple (k) of the fundamental frequency ω_0, yielding zero for all k. Hence:

$$X_k = -\frac{A}{T_0 jk\omega_0}\left(\frac{T_0}{2}e^{-jk\omega_0\frac{T_0}{2}} + \frac{T_0}{2}e^{jk\omega_0\frac{T_0}{2}}\right)$$

Using $\omega_0 T_0 = 2\pi$ and $\omega_0\frac{T_0}{2} = \pi$, we obtain:

$$X_k = -\frac{A}{jk2\pi}\frac{T_0}{2}\left(e^{-jk\pi} + e^{jk\pi}\right)$$

Then, using $\frac{1}{2}\left(e^{-j\theta} + e^{j\theta}\right) = \cos\theta$, (C.2), and $\frac{T_0}{2\pi} = \frac{1}{\omega_0}$, we obtain:

$$X_k = -\frac{A}{jk\omega_0}\cos(k\pi)$$

Note that $\cos(k\pi) = (-1)^k$:

$$X_k = -\frac{A}{jk\omega_0}(-1)^k = j\frac{A}{k\omega_0}(-1)^k \quad \text{for all } k \text{ except } k = 0$$

Computing X_0 separately:

$$X_0 = \frac{A}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} te^{-j0\omega_0 t}\,\mathrm{d}t = \frac{A}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} t\,\mathrm{d}t = \frac{A}{2T_0}\left[t^2\right]_{-\frac{T_0}{2}}^{\frac{T_0}{2}} = 0$$

$X_0 = 0$ could also be obtained from the illustration of the waveform, which clearly shows that the signal has zero mean.

Concluding: $$X_k = \begin{cases} j\frac{A}{k\omega_0}(-1)^k & \text{for all } k \text{ except } k = 0 \\ 0 & \text{for } k = 0 \end{cases}$$

As a quick sanity check: note that from the illustration of the waveform, we can already see that this (real) signal is odd. Hence, its complex exponential Fourier series coefficients have zero real parts and are all imaginary. Indeed, that is what we have just derived above.

Method 2: Compute trigonometric Fourier series coefficients and put them together

Again, from the illustration of the waveform, we can already see that this (real) signal is odd. This means that all a_k coefficients are zero, including a_0. Hence, we only need the sines. Let us start with computing the b_k's using (3.5):

$$b_k = \frac{2}{T_0}\int_{T_0} x(t)\sin(k\omega_0 t)\,\mathrm{d}t = \frac{2A}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} t\sin(k\omega_0 t)\,\mathrm{d}t$$

Using the partial integration rule with $\frac{\mathrm{d}f(t)}{\mathrm{d}t} = \sin(k\omega_0 t)$ and $g(t) = t$, from which it follows that $f(t) = -\frac{1}{k\omega_0}\cos(k\omega_0 t)$ and $\frac{\mathrm{d}g(t)}{\mathrm{d}t} = 1$:

$$b_k = -\frac{2A}{T_0 k\omega_0}\left(\left[t\cos(k\omega_0 t)\right]_{-\frac{T_0}{2}}^{\frac{T_0}{2}} - \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}}\cos(k\omega_0 t)\,\mathrm{d}t\right)$$

The remaining integral on the right-hand side equals zero for all indices k, as we integrate a cosine that fits exactly k times in the period T_0 of integration.

$$b_k = -\frac{2A}{T_0 k\omega_0}\left(\frac{T_0}{2}\cos\left(k\omega_0\frac{T_0}{2}\right) - \frac{-T_0}{2}\cos\left(k\omega_0\frac{-T_0}{2}\right)\right)$$

Using $\omega_0\frac{T_0}{2} = \pi$:

$$b_k = -\frac{2A}{T_0 k\omega_0}\left(\frac{T_0}{2}\cos(k\pi) + \frac{T_0}{2}\cos(-k\pi)\right) = -\frac{2A}{k\omega_0}\cos(k\pi)$$

because the cosine is even. The remaining cosine function equals 1 for even indices k and -1 for odd indices k. Then b_k can be simplified to:

$$b_k = -\frac{2A}{k\omega_0}(-1)^k \quad \text{with} \quad k \in \mathbb{N}^+$$

Now, let us convert these coefficients into the complex exponential Fourier series coefficients using (4.4):

$$X_0 = a_0 \qquad \text{and} \qquad X_k = \frac{1}{2}(a_k - jb_k) \qquad \text{and} \qquad X_{-k} = \frac{1}{2}(a_k + jb_k) \qquad \text{for} \quad k \in \mathbb{N}^+$$

Concluding: $$X_k = \begin{cases} j\frac{A}{k\omega_0}(-1)^k & \text{for all } k \text{ except } k = 0 \\ 0 & \text{for } k = 0 \end{cases}$$

This is the same answer as that obtained using Method 1, as it should be.

S.4-11

(a) Note that this signal equals A from $t = t_0 - \frac{\tau}{2}$ to $t = t_0 + \frac{\tau}{2}$ and zero in the remainder of period T_0. Hence, using the analysis equation (4.3):

$$X_k = \frac{1}{T_0}\int_{T_0} x(t)e^{-jk\omega_0 t}\,\mathrm{d}t = \frac{1}{T_0}\int_{t_0-\frac{\tau}{2}}^{t_0+\frac{\tau}{2}} Ae^{-jk\omega_0 t}\,\mathrm{d}t = \left[-\frac{A}{jk\omega_0 T_0}e^{-jk\omega_0 t}\right]_{t_0-\frac{\tau}{2}}^{t_0+\frac{\tau}{2}}$$

$$= -\frac{A}{jk\omega_0 T_0}\left(e^{-jk\omega_0(t_0+\frac{\tau}{2})} - e^{-jk\omega_0(t_0-\frac{\tau}{2})}\right) = \frac{2A}{k\omega_0 T_0}j\frac{1}{2}e^{-jk\omega_0 t_0}\left(e^{-jk\omega_0\frac{\tau}{2}} - e^{jk\omega_0\frac{\tau}{2}}\right)$$

Note that the above equation only holds for $k \neq 0$. When first computing X_0:

$$X_0 = \frac{1}{T_0}\int_{t_0-\frac{\tau}{2}}^{t_0+\frac{\tau}{2}} Ae^{-j0\omega_0 t}\,\mathrm{d}t = \frac{A}{T_0}\int_{t_0-\frac{\tau}{2}}^{t_0+\frac{\tau}{2}} 1\,\mathrm{d}t = \frac{A}{T_0}[t]_{t_0-\frac{\tau}{2}}^{t_0+\frac{\tau}{2}} = \frac{A\tau}{T_0}$$

Then, using $j\frac{1}{2}\left(e^{-j\theta} - e^{j\theta}\right) = \sin\theta$, (C.3), and substituting $\omega_0 = \frac{2\pi}{T_0}$ into the earlier equation for X_k:

$$X_k = \frac{2A}{k\omega_0 T_0}e^{-jk\omega_0 t_0}\sin\left(k\omega_0\frac{\tau}{2}\right) = \frac{A}{k\pi}e^{-jk\omega_0 t_0}\sin\left(\frac{\pi k\tau}{T_0}\right) = \frac{A\tau}{T_0}e^{-jk\omega_0 t_0}\frac{\sin\left(\frac{\pi k\tau}{T_0}\right)}{\frac{\pi k\tau}{T_0}}$$

where the numerator and denominator have been multiplied by $\frac{\pi k\tau}{T_0}$.

Finally, using the sinc function defined in (B.3):

$$X_k = \frac{A\tau}{T_0} e^{-jk\omega_0 t_0} \operatorname{sinc}\left(\frac{k\tau}{T_0}\right)$$

When filling in $k = 0$:

$$X_0 = \frac{A\tau}{T_0} e^{-j0\omega_0 t_0} \operatorname{sinc}\left(\frac{0\tau}{T_0}\right) = \frac{A\tau}{T_0} \qquad \text{where we used } \operatorname{sinc}(0) = 1.$$

This is the same answer for X_0 as that obtained above, as it should be.

Note that this signal is very similar to the signal in Example 4.3, with the only difference being the time shift $-t_0$ (time shift to the *right*) included in this exercise. In Example 4.3 it was already shown that for a pulse train signal $x(t)$ *without* time shift $-t_0$ the complex exponential Fourier series coefficients X_k equal $\frac{A\tau}{T_0} \operatorname{sinc}\left(\frac{k\tau}{T_0}\right)$. Using this result and applying the time shift theorem (4.13) from Section 4.6, one would readily arrive at the complex exponential Fourier series coefficients X_k derived above for a pulse train signal $x(t)$ *with* time shift $-t_0$.

(b) Whereas, for the sake of completeness, below we will show how to obtain the complex exponential Fourier series coefficients the 'straightforward' way using the analysis equation (4.3), a much quicker way would be to note that the time derivative of triangular wave $x(t)$ is an odd square wave $y(t)$ with amplitude $\frac{4A}{T_0}$ from $t = -\frac{T_0}{2}$ to $t = 0$ s and $-\frac{4A}{T_0}$ from $t = 0$ to $t = \frac{T_0}{2}$ s, then compute the complex exponential Fourier series coefficients Y_k of this odd square wave $y(t)$ and finally, from that, work towards coefficients X_k for triangular wave $x(t)$ using the differentiation theorem (4.15) from Section 4.6. Both methods lead to the same result. Of course, since the triangular wave's real, or trigonometric, Fourier series coefficients were already obtained in Exercise *e*.3-4(b), one could also take those as a starting point and then put them together to obtain the complex exponential Fourier series coefficients X_k.

Method 1: Direct calculation of complex exponential Fourier series coefficients

Using the analysis equation (4.3):

$$X_k = \frac{1}{T_0}\int_{T_0} x(t)e^{-jk\omega_0 t}\,\mathrm{d}t = \frac{1}{T_0}\left(\int_{-\frac{T_0}{2}}^{0}\left(A + \frac{4A}{T_0}t\right) e^{-jk\omega_0 t}\,\mathrm{d}t + \int_0^{\frac{T_0}{2}}\left(A - \frac{4A}{T_0}t\right) e^{-jk\omega_0 t}\,\mathrm{d}t\right)$$

$$= \frac{1}{T_0}\left(\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} Ae^{-jk\omega_0 t}\,\mathrm{d}t + \frac{4A}{T_0}\int_{-\frac{T_0}{2}}^{0} te^{-jk\omega_0 t}\,\mathrm{d}t - \frac{4A}{T_0}\int_0^{\frac{T_0}{2}} te^{-jk\omega_0 t}\,\mathrm{d}t\right)$$

We then use the partial integration rule for the second and third integral that states that $\int_a^b \frac{\mathrm{d}f(x)}{\mathrm{d}x} g(x)\,\mathrm{d}x = [f(x)g(x)]_a^b - \int_a^b f(x)\frac{\mathrm{d}g(x)}{\mathrm{d}x}\,\mathrm{d}x$: define $\frac{\mathrm{d}f(t)}{\mathrm{d}t} = e^{-jk\omega_0 t}$ and $g(t) = t$, from which it follows that $f(t) = -\frac{1}{jk\omega_0}e^{-jk\omega_0 t}$ and $\frac{\mathrm{d}g(t)}{\mathrm{d}t} = 1$, and we obtain:

$$X_k = \frac{A}{T_0}\left(\left[\frac{1}{-jk\omega_0}e^{-jk\omega_0 t}\right]_{-\frac{T_0}{2}}^{\frac{T_0}{2}} - \frac{4}{T_0 jk\omega_0}\left(\left[te^{-jk\omega_0 t}\right]_{-\frac{T_0}{2}}^{0} - \int_{-\frac{T_0}{2}}^{0} e^{-jk\omega_0 t}\,\mathrm{d}t\right)\right.$$
$$\left. + \frac{4}{T_0 jk\omega_0}\left(\left[te^{-jk\omega_0 t}\right]_0^{\frac{T_0}{2}} - \int_0^{\frac{T_0}{2}} e^{-jk\omega_0 t}\,\mathrm{d}t\right)\right)$$

$$= \frac{A}{T_0}\left(\left[\frac{1}{-jk\omega_0}e^{-jk\omega_0 t}\right]_{-\frac{T_0}{2}}^{\frac{T_0}{2}} - \frac{4}{T_0 jk\omega_0}\left(\left[te^{-jk\omega_0 t}\right]_{-\frac{T_0}{2}}^{0} - \left[\frac{1}{-jk\omega_0}e^{-jk\omega_0 t}\right]_{-\frac{T_0}{2}}^{0}\right)\right.$$
$$\left. + \frac{4}{T_0 jk\omega_0}\left(\left[te^{-jk\omega_0 t}\right]_0^{\frac{T_0}{2}} - \left[\frac{1}{-jk\omega_0}e^{-jk\omega_0 t}\right]_0^{\frac{T_0}{2}}\right)\right)$$

Using $\omega_0 = \frac{2\pi}{T_0}$ we obtain:

$$X_k = -\frac{A}{jk2\pi}\left(e^{-jk\pi} - e^{jk\pi} + 2e^{jk\pi} + \frac{2}{jk\pi}\left(1 - e^{jk\pi}\right) - 2e^{-jk\pi} - \frac{2}{jk\pi}\left(e^{-jk\pi} - 1\right)\right)$$

Both $e^{-jk\pi}$ and $e^{jk\pi}$ equal 1 for even indices k (including $k = 0$) and -1 for odd indices k. Therefore, both $\left(e^{-jk\pi} - e^{jk\pi}\right)$ and $2\left(e^{jk\pi} - e^{-jk\pi}\right)$ equal zero for all indices k.
Then we obtain:

$$X_k = -\frac{A}{jk2\pi}\left(\frac{4}{jk\pi} - \frac{2}{jk\pi}\left(e^{jk\pi} + e^{-jk\pi}\right)\right) = \frac{A}{k^2\pi^2}\left(2 - e^{jk\pi} - e^{-jk\pi}\right)$$

Again, since both $e^{-jk\pi}$ and $e^{jk\pi}$ equal 1 for even indices k and -1 for odd indices k:

$$X_k = \begin{cases} \frac{4A}{k^2\pi^2} & \text{for } k \text{ odd} \\ 0 & \text{for } k \text{ even (including } k = 0) \end{cases}$$

As a quick sanity check: note that from the illustration of the waveform, we can already see that this (real) signal is even. Hence, its complex exponential Fourier series coefficients are all real and even. Furthermore, the signal also has half-wave odd symmetry and a zero mean. Hence, all even-indexed coefficients (including $k = 0$) are zero. Indeed, the above-derived coefficients are zero for k even. Furthermore, for k odd, all coefficients are real (imaginary parts equal zero). They are also even, since k appears as k^2. Hence, for positive and negative k, we obtain the same values.

Method 2: Apply the differentiation theorem to the coefficients of an odd square wave

Note that this method is related to Exercises e.4-12 to e.4-14 from Section 4.6.
Consider the odd square wave $y(t)$, which is the first (time-)derivative of the triangular wave $x(t)$ defined in this exercise. Then, $y(t) = \frac{\mathrm{d}}{\mathrm{d}t}x(t)$. This derivative $y(t)$ is illustrated in Solution s.4-14. It has amplitude $B = \frac{4A}{T_0}$. The odd square wave $y(t)$ has the following complex exponential Fourier series coefficients:

$$Y_k = \frac{1}{T_0}\int_{T_0} x(t)e^{-jk\omega_0 t}\,\mathrm{d}t = \frac{1}{T_0}\left(\int_{-\frac{T_0}{2}}^{0} Be^{-jk\omega_0 t}\,\mathrm{d}t + \int_{0}^{\frac{T_0}{2}} (-B)e^{-jk\omega_0 t}\,\mathrm{d}t\right)$$

$$= \frac{B}{T_0}\left(\left[\frac{1}{-jk\omega_0}e^{-jk\omega_0 t}\right]_{-\frac{T_0}{2}}^{0} - \left[\frac{1}{-jk\omega_0}e^{-jk\omega_0 t}\right]_{0}^{\frac{T_0}{2}}\right)$$

Using $\omega_0 = \frac{2\pi}{T_0}$:

$$Y_k = -\frac{B}{T_0}\left(\left[\frac{1}{jk\frac{2\pi}{T_0}}e^{-jk\frac{2\pi}{T_0}t}\right]_{-\frac{T_0}{2}}^{0} - \left[\frac{1}{jk\frac{2\pi}{T_0}}e^{-jk\frac{2\pi}{T_0}t}\right]_{0}^{\frac{T_0}{2}}\right)$$

$$= -\frac{B}{jk2\pi}\left(1 - e^{jk\frac{2\pi}{T_0}\frac{T_0}{2}} - e^{-jk\frac{2\pi}{T_0}\frac{T_0}{2}} + 1\right) = -\frac{B}{jk2\pi}\left(2 - e^{jk\pi} - e^{-jk\pi}\right)$$

Since both $e^{jk\pi}$ and $e^{-jk\pi}$ equal 1 for even indices k (including $k = 0$) and -1 for odd indices k:

$$Y_k = \begin{cases} j\frac{2B}{k\pi} & \text{for } k \text{ odd} \\ 0 & \text{for } k \text{ even (including } k = 0) \end{cases}$$

Or even faster: take the complex exponential Fourier series coefficients for an even square wave, as obtained in Exercise e.4-10(c), and apply the time shift theorem (4.13) defined in Section 4.6 to obtain coefficients Y_k for the odd square wave $y(t)$. Just use amplitude B for the even square wave in this case instead of A, which was the amplitude of the even square wave from Exercise e.4-10(c). The coefficients found in Solution s.4-10(c), but here defined for amplitude B instead of A, are as follows:

$$X_{k,\text{even-sq-wave}} = \begin{cases} \frac{2B}{|k|\pi} & \text{for } k = \pm 1, \pm 5, \pm 9, \ldots \\ -\frac{2B}{|k|\pi} & \text{for } k = \pm 3, \pm 7, \pm 11, \ldots \\ 0 & \text{for } k \text{ even (including } k = 0) \end{cases}$$

From the time shift theorem (4.13) defined in Section 4.6, we know that:

$$Y_k = X_{k,\text{even-sq-wave}} e^{jk\omega_0 t_0} \qquad \text{with} \qquad t_0 = \frac{T_0}{4}$$

Then, we obtain for the complex exponential Fourier series coefficients of an odd square wave $y(t)$ with amplitude B:

$$Y_k = \begin{cases} j\frac{2B}{k\pi} & \text{for } k \text{ odd} \\ 0 & \text{for } k \text{ even (including } k = 0) \end{cases}$$

where we used the fact that $e^{jk\frac{2\pi}{T_0}\frac{T_0}{4}} = e^{jk\frac{\pi}{2}}$ equals j for $k = 1, 5, 9, \ldots$ and $-j$ for $k = -1, -5, -9, \ldots$; also, it equals $-j$ for $k = 3, 7, 11, \ldots$ and j for $k = -3, -7, -11, \ldots$.

Substituting $B = \frac{4A}{T_0}$ for the amplitude:

$$Y_k = \begin{cases} j\frac{8A}{k\pi T_0} & \text{for } k \text{ odd} \\ 0 & \text{for } k \text{ even (including } k = 0) \end{cases}$$

Now because $y(t) = \frac{\mathrm{d}}{\mathrm{d}t}x(t)$, we can use the differentiation theorem, $Y_k = (jk\omega_0)X_k$, (4.15), to obtain $X_k = \frac{Y_k}{jk\omega_0}$. Concluding, because $\frac{1}{jk\omega_0} j\frac{8A}{k\pi T_0} = \frac{4A}{k^2\pi^2}$, where we used $\omega_0 = \frac{2\pi}{T_0}$:

$$X_k = \begin{cases} \frac{4A}{k^2\pi^2} & \text{for } k \text{ odd} \\ 0 & \text{for } k \text{ even (including } k = 0) \end{cases}$$

This is the same answer as that obtained using Method 1, as it should be. However, this method does require the differentiation theorem (4.15), and optionally, also the time shift theorem (4.13), which are only defined in upcoming Section 4.6. Hence, one should have studied this upcoming section already to be able to apply Method 2. Methods 1 and 3, however, can be applied without requiring any material from upcoming sections.

Method 3: Compute trigonometric Fourier series coefficients and put them together

From Exercise *e*.3-4(b) (see Solution *s*.3-4(b)), we know that the real, or trigonometric, Fourier series coefficients of the triangular wave $x(t)$ are:

$$a_k = \begin{cases} \frac{8A}{k^2\pi^2} & \text{for } k \text{ odd} \\ 0 & \text{for } k \text{ even (including } k = 0) \end{cases} \qquad \text{all } b_k = 0$$

Now, let us convert these coefficients into the complex exponential Fourier series coefficients using (4.4):

$$X_0 = a_0 \qquad \text{and} \qquad X_k = \tfrac{1}{2}(a_k - jb_k) \qquad \text{and} \qquad X_{-k} = \tfrac{1}{2}(a_k + jb_k) \qquad \text{for } k \in \mathbb{N}^+$$

We directly obtain the desired result by substitution of the a_k's and b_k's in the above:

$$X_k = \begin{cases} \frac{4A}{k^2\pi^2} & \text{for } k \text{ odd} \\ 0 & \text{for } k \text{ even (including } k = 0) \end{cases}$$

This is the same answer as that obtained using Methods 1 and 2, as it should be.

S | 4.6 Two Fourier series theorems

S.4-12

Suppose the complex exponential Fourier series coefficients of signal $x(t)$ equal X_k. Define waveform $y(t) = x(t + t_0)$ which is a time-shifted copy of $x(t)$ for time shift $t_0 \in \mathbb{R}$. Then:

$$Y_k = \frac{1}{T_0}\int_{T_0} y(t)e^{-jk\omega_0 t}\,\mathrm{d}t = \frac{1}{T_0}\int_{T_0} x(t+t_0)e^{-jk\omega_0 t}\,\mathrm{d}t$$

Define variable $t' = t + t_0$, then $t = t' - t_0$ and $\mathrm{d}t' = \mathrm{d}t$ (as t_0 is a constant). We get:

$$Y_k = \frac{1}{T_0}\int_{T_0} x(t')e^{-jk\omega_0 (t'-t_0)}\,\mathrm{d}t' = \frac{1}{T_0}\int_{T_0} x(t')e^{-jk\omega_0 t'}e^{jk\omega_0 t_0}\,\mathrm{d}t'$$
$$= \left(\frac{1}{T_0}\int_{T_0} x(t')e^{-jk\omega_0 t'}\,\mathrm{d}t'\right)e^{jk\omega_0 t_0} = X_k e^{jk\omega_0 t_0} \quad \text{with} \;\; k \in \mathbb{Z}$$

where in the last step we substituted $t' = t$. We obtain:

$$Y_k = X_k e^{jk\omega_0 t_0} \quad \text{with} \;\; k \in \mathbb{Z}$$

qed

S.4-13

The partial integration rule states that $\int_a^b \frac{\mathrm{d}f(x)}{\mathrm{d}x}g(x)\,\mathrm{d}x = [f(x)g(x)]_a^b - \int_a^b f(x)\frac{\mathrm{d}g(x)}{\mathrm{d}x}\,\mathrm{d}x$. So when defining $\frac{\mathrm{d}f(t)}{\mathrm{d}t} = \frac{\mathrm{d}x(t)}{\mathrm{d}t}$, and $g(t) = e^{-jk\omega_0 t}$, from which it follows that $f(t) = x(t)$ and $\frac{\mathrm{d}g(t)}{\mathrm{d}t} = (-jk\omega_0)e^{-jk\omega_0 t}$, we obtain the following when integrating by parts:

$$T_0 X_k' = \int_{T_0} \frac{\mathrm{d}x(t)}{\mathrm{d}t}e^{-jk\omega_0 t}\,\mathrm{d}t = \left[x(t)e^{-jk\omega_0 t}\right]_0^{T_0} - \int_{T_0} x(t)(-jk\omega_0)e^{-jk\omega_0 t}\,\mathrm{d}t$$
$$= \left(x(T_0)e^{-jk\omega_0 T_0}\right) - \left(x(0)\cdot 1\right) + (jk\omega_0)\int_{T_0} x(t)e^{-jk\omega_0 t}\,\mathrm{d}t$$

Using $\omega_0 = \frac{2\pi}{T_0}$ and the analysis equation, $\int_{T_0} x(t)e^{-jk\omega_0 t}\,\mathrm{d}t = T_0 X_k$, (4.3):

$$T_0 X_k' = x(T_0)e^{-jk2\pi} - x(0) + (jk\omega_0)T_0 X_k = x(T_0) - x(0) + (jk\omega_0)T_0 X_k$$

where we made use of the fact that $e^{-jk2\pi} = 1 \quad \forall \;\; k \in \mathbb{Z}$.
When also noting that $x(T_0) - x(0) = 0$ as a result of periodicity, we obtain:

$$T_0 X_k' = (jk\omega_0)T_0 X_k$$

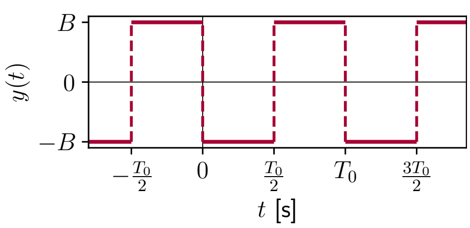
Concluding:

$$X_k' = (jk\omega_0)X_k$$

Integration by parts can be applied repeatedly to prove (4.15) for n differentiations. **qed**

S.4-14

Let us call the triangular wave from Exercise E.4-11(b) function $x(t)$. When taking the first (time-)derivative of $x(t)$, we obtain the odd-symmetry square wave $y(t)$, illustrated here on the right. Its amplitude B can be easily computed: $B = \frac{2A}{\frac{T_0}{2}} = \frac{4A}{T_0}$.



We see that $y(t) = \frac{\mathrm{d}}{\mathrm{d}t}x(t)$ is very similar to the even square wave from Exercise E.4-10(c), with the exception that the even square wave has amplitude A, rather than B, and that the odd square wave $y(t)$ is a time-shifted version of the even square wave. Let us call this even square wave from Exercise E.4-10(c) function $z(t)$ and describe it with amplitude B. Then:

$$y(t) = z(t + \tfrac{T_0}{4})$$

In Exercise E.4-11(b), we have computed the complex exponential Fourier series coefficients of the triangular wave $x(t)$ (see Solution S.4-11(b)):

$$X_k = \begin{cases} \frac{4A}{k^2\pi^2} & \text{for } k \text{ odd} \\ 0 & \text{for } k \text{ even} \end{cases}$$

Note that all even-indexed complex exponential Fourier series coefficients (including $k = 0$) are zero, which is a consequence of the fact that the triangular wave $x(t)$ has half-wave odd symmetry. Now because $y(t) = \frac{\mathrm{d}}{\mathrm{d}t}x(t)$, we can use the differentiation theorem, $Y_k = (jk\omega_0)X_k$, (4.15), to obtain Y_k. Concluding, because $\left(jk\frac{2\pi}{T_0}\right)\frac{4A}{k^2\pi^2} = j\frac{8A}{k\pi T_0}$ and $A = \frac{BT_0}{4}$:

$$Y_k = \begin{cases} j\frac{2B}{k\pi} & \text{for } k \text{ odd} \\ 0 & \text{for } k \text{ even} \end{cases}$$

The exercise ends here. We are done.

However, let us check the above result using the complex exponential Fourier series coefficients of the above-mentioned even square wave $z(t)$, as we have computed those in Exercise e.4-10(c) (see Solution s.4-10(c)):

$$Z_k = \begin{cases} \frac{2B}{|k|\pi} & \text{for } k = \pm1, \pm5, \pm9, \dots \\ -\frac{2B}{|k|\pi} & \text{for } k = \pm3, \pm7, \pm11, \dots \\ 0 & \text{for } k \text{ even (including } k = 0) \end{cases}$$

Remember that we now use B as the amplitude of $z(t)$, not A.

Since $y(t) = z(t + \frac{T_0}{4})$, we can apply the time shift theorem (4.13) to obtain the complex exponential Fourier series coefficients Y_k:

$$Y_k = Z_k e^{jk\omega_0\frac{T_0}{4}} = Z_k e^{jk\frac{\pi}{2}} \qquad (*)$$

To rapidly move from Z_k to Y_k using relation (*), just make a table (see on the right). Note that all even-indexed coefficients remain to be zero under (*). Further note that only a small subset of k's is needed to 'get an idea of what is happening'. Finally, note that when dealing with real-valued signals, as we do in this book, the complex exponential Fourier series coefficients X_{-k} and X_k (for $k \in \mathbb{N}^+$) should always be each other's *complex conjugate*. As can be deduced from the table:

$$Y_k = \begin{cases} j\frac{2B}{k\pi} & \text{for } k \text{ odd} \\ 0 & \text{for } k \text{ even} \end{cases}$$

These coefficients are identical to those obtained earlier. Applying the differentiation theorem for the complex exponential Fourier series (4.15) indeed yields the correct result.

k	$e^{jk\frac{\pi}{2}}$	Z_k	Y_k
-7	j	$-\frac{2B}{7\pi}$	$-j\frac{2B}{7\pi}$
-5	$-j$	$\frac{2B}{5\pi}$	$-j\frac{2B}{5\pi}$
-3	j	$-\frac{2B}{3\pi}$	$-j\frac{2B}{3\pi}$
-1	$-j$	$\frac{2B}{\pi}$	$-j\frac{2B}{1\pi}$
1	j	$\frac{2B}{\pi}$	$j\frac{2B}{1\pi}$
3	$-j$	$-\frac{2B}{3\pi}$	$j\frac{2B}{3\pi}$
5	j	$\frac{2B}{5\pi}$	$j\frac{2B}{5\pi}$
7	$-j$	$-\frac{2B}{7\pi}$	$j\frac{2B}{7\pi}$

s.4-15

From the illustration of signal $x(t)$, it follows that:

$$T_0 = 4 \text{ s}, \quad \text{hence} \quad f_0 = \frac{1}{T_0} = \frac{1}{4} \text{ Hz} \quad \text{and} \quad \omega_0 = 2\pi f_0 = 2\pi\frac{1}{4} = \frac{\pi}{2} \text{ rad/s}$$

It can also be deduced from the exercise that:

$$X_{-2} = 0, \quad X_{-1} = -j\frac{2}{k\pi} = -j\frac{2}{-1\pi} = j\frac{2}{\pi}, \quad X_0 = -\frac{1}{2}, \quad X_1 = -j\frac{2}{k\pi} = -j\frac{2}{\pi} \quad \text{and} \quad X_2 = 0$$

Using the differentiation theorem (4.15) for the second time-derivative of $x(t)$:

$$\begin{aligned}
Y_k = (jk\omega_0)^2 X_k:\ \ Y_0 &= \left(jk\tfrac{\pi}{2}\right)^2 X_0 = \left(j(0)\tfrac{\pi}{2}\right)^2\left(-\tfrac{1}{2}\right) = 0\\
Y_2 &= \left(jk\tfrac{\pi}{2}\right)^2 X_2 = \left(j(2)\tfrac{\pi}{2}\right)^2(0) = 0\\
Y_{-2} &= \left(jk\tfrac{\pi}{2}\right)^2 X_{-2} = \left(j(-2)\tfrac{\pi}{2}\right)^2(0) = 0 \quad \text{or} \quad Y_{-2} = Y_2^* = 0\\
Y_1 &= \left(jk\tfrac{\pi}{2}\right)^2 X_1 = \left(j(1)\tfrac{\pi}{2}\right)^2\left(-j\tfrac{2}{\pi}\right) = \left(-\tfrac{1}{4}\pi^2\right)\left(-j\tfrac{2}{\pi}\right) = j\tfrac{\pi}{2}\\
Y_{-1} &= \left(jk\tfrac{\pi}{2}\right)^2 X_{-1} = \left(j(-1)\tfrac{\pi}{2}\right)^2\left(j\tfrac{2}{\pi}\right) = -j\tfrac{\pi}{2} \quad \text{or} \quad Y_{-1} = Y_1^* = -j\tfrac{\pi}{2}
\end{aligned}$$

Hence, the complex exponential Fourier series coefficients Y_k are:

k	-2	-1	0	1	2
Y_k	0	$-j\frac{\pi}{2}$	0	$j\frac{\pi}{2}$	0

S.4-16

(a) Using the synthesis equation, $x(t) = \sum_{k=-\infty}^{\infty} X_k e^{jk\omega_0 t}$, (4.2), it can be found that:

$$y(t) = x(t-\tau) = \sum_{k=-\infty}^{\infty} X_k e^{jk\omega_0(t-\tau)} = \sum_{k=-\infty}^{\infty} X_k e^{-jk\omega_0\tau} e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} Y_k e^{jk\omega_0 t}$$

Concluding:

$$Y_k = X_k e^{-jk\omega_0\tau}$$

Note that this is the time shift theorem defined in (4.13) with a negative time shift.

(b) From the analysis equation, $X_k = \frac{1}{T_0}\int_{T_0} x(t)e^{-jk\omega_0 t}\,\mathrm{d}t$, (4.3), we obtain:

$$X_{k-1} = \frac{1}{T_0}\int_{T_0} x(t)e^{-j(k-1)\omega_0 t}\,\mathrm{d}t$$

We also obtain:

$$Y_k = \frac{1}{T_0}\int_{T_0} y(t)e^{-jk\omega_0 t}\,\mathrm{d}t = \frac{1}{T_0}\int_{T_0} e^{j2\pi f_0 t}x(t)e^{-jk\omega_0 t}\,\mathrm{d}t = \frac{1}{T_0}\int_{T_0} x(t)e^{-j(k\omega_0 t - 2\pi f_0 t)}\,\mathrm{d}t$$

Using $\omega_0 = 2\pi f_0$:

$$Y_k = \frac{1}{T_0}\int_{T_0} x(t)e^{-j(k\omega_0 t - \omega_0 t)}\,\mathrm{d}t = \frac{1}{T_0}\int_{T_0} x(t)e^{-j(k-1)\omega_0 t}\,\mathrm{d}t = X_{k-1}$$

Concluding:

$$Y_k = X_{k-1}$$

S | 4.7 Parseval: complex exponential Fourier series

S.4-17

The average power of a (real) periodic signal $x(t)$ was defined in (2.19): $P = \frac{1}{T_0}\int_{T_0} x(t)x(t)\,\mathrm{d}t$
Using the synthesis equation (4.2):

$$P = \sum_{k=-\infty}^{\infty} X_k \frac{1}{T_0}\int_{T_0} x(t)e^{jk\omega_0 t}\,\mathrm{d}t = \sum_{k=-\infty}^{\infty} X_k \left(\frac{1}{T_0}\int_{T_0} x(t)e^{-jk\omega_0 t}\,\mathrm{d}t\right)^*$$

where we made use of the fact that signal $x(t)$ is real ($x^*(t) = x(t)$).
Using the analysis equation (4.3) the part between brackets is replaced by X_k and we obtain:

$$P = \sum_{k=-\infty}^{\infty} X_k X_k^* = \sum_{k=-\infty}^{\infty} |X_k|^2 = X_0^2 + 2\sum_{k=1}^{\infty} |X_k|^2$$

qed

5

Fourier transform

Exercises

e|5.4 Interpretation, polar form

e.5-1

If the Fourier transform of a real signal $x(t)$ is expressed in polar form as

$$X(f) = |X(f)|\, e^{j\angle X(f)} \tag{5.5}$$

show that

$$|X(-f)| = |X(f)|$$
$$\angle X(-f) = -\angle X(f)$$

i.e., the magnitude is even and the phase is odd.

e|5.5 Effects of symmetry

e.5-2

(a) Show that if $x(t)$ is real and even, i.e., $x(t) = x(-t)$, then the Fourier transform of $x(t)$ may be reduced to

$$X(f) = 2\int_0^\infty x(t)\cos(2\pi f t)\,\mathrm{d}t$$

Therefore, conclude that $X(f)$ is real and even if $x(t)$ is real and even.

(b) Show that if $x(t)$ is real and odd, i.e., $x(t) = -x(-t)$, then the Fourier transform of $x(t)$ may be reduced to

$$X(f) = -2j\int_0^\infty x(t)\sin(2\pi f t)\,\mathrm{d}t$$

Therefore, conclude that $X(f)$ is imaginary and odd if $x(t)$ is real and odd.

e.5-3

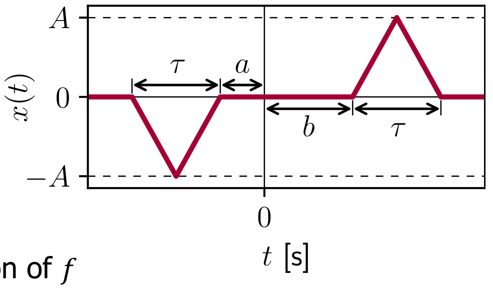
Consider the signal $x(t)$ shown on the right. Fourier transform $X(f)$ of this signal can be computed.

Determine for each of the following statements whether the statement is true or false.

(I) $X(0)$ equals 0

(II) $x(t)$ has half-wave odd symmetry

(III) If $a = b$, $X(f)$ will be a purely imaginary function of f



e.5-4 Consider the signal $x(t)$ shown on the right. Fourier transform $X(f)$ of this signal can be computed.

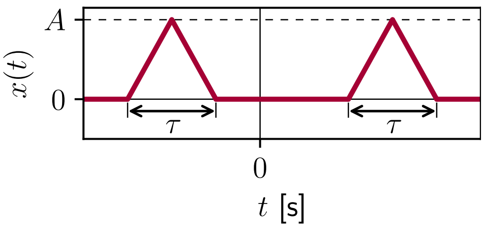
Which of the following statements is correct?

A. $X(0)$ equals $A\tau$

B. $X(f)$ will be a purely imaginary function of f

C. $X(f)$ will be a purely real function of f

D. None of the other statements is correct



e|5.7 Examples

e.5-5 Compute the Fourier transforms of the following signals (with $\alpha > 0$, if α included):

(a) $x_a(t) = te^{-\alpha t}u(t)$

(b) $x_b(t) = At^2u(t)u(1-t)$

(c) $x_c(t) = Ae^{-\alpha t}u(t)u(1-t)$

(d) $x_d(t) = e^{-\pi\left(\frac{t}{\tau}\right)^2}$

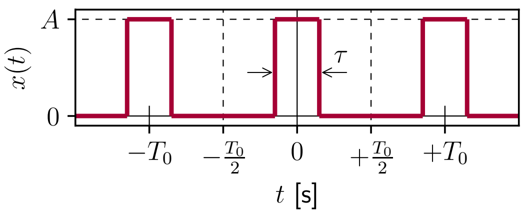
Hint: for (d), use the definite integral $\int_{-\infty}^{\infty} e^{-a^2x^2+bx}\,\mathrm{d}x = \frac{\sqrt{\pi}}{a}e^{\frac{b^2}{4a^2}}$, with $a > 0$.

e.5-6 Obtain time signal $x(t)$ corresponding to the Fourier transform

$$X(f) = Ae^{-\alpha|f|} \qquad \text{for } \alpha > 0.$$

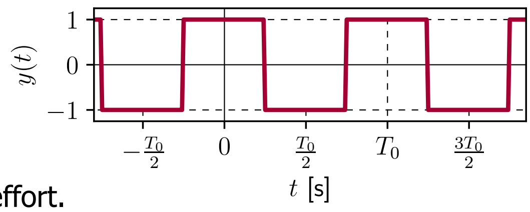
e.5-7 Show by Fourier-transforming one period T_0 of periodic pulse train signal $x(t)$ illustrated on the right, that the complex exponential Fourier series coefficients X_k of this signal $x(t)$ are:

$$X_k = \frac{A\tau}{T_0}\operatorname{sinc}\left(k\frac{\tau}{T_0}\right) \quad \text{with } k \in \mathbb{Z} \qquad (4.12)$$



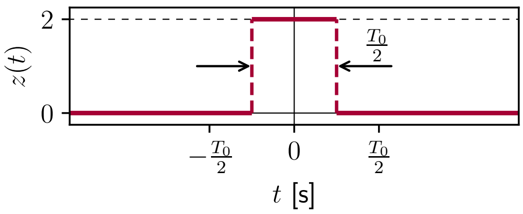
e.5-8 In Example 4.4, the complex exponential Fourier series coefficients of the periodic even square wave $y(t)$ illustrated below on the right, were computed. The complex exponential Fourier series coefficients were found to be:

$$Y_k = \begin{cases} +\frac{2}{|k|\pi} & \text{for } k = \pm1, \pm5, \pm9, \ldots \\ -\frac{2}{|k|\pi} & \text{for } k = \pm3, \pm7, \pm11, \ldots \\ 0 & \text{for } k \text{ even (including } k = 0) \end{cases}$$



Computing these coefficients required substantial effort.
In this exercise, we explore a more convenient approach to obtain the same result.

Start by Fourier-transforming pulse function $z(t)$ shown on the right. Describe how pulse function $z(t)$ differs from a single period of the even square wave $y(t)$ illustrated above on the right. Work from there to obtain the complex exponential Fourier series coefficients Y_k of $y(t)$.



e|5.8 Fourier transform in-the-limit

e.5-9

Compute Fourier transform $X(f)$ of the signal

$$x(t) = \frac{1}{4}\Big(\delta(t+1) + \delta\big(t+\tfrac{1}{2}\big) + \delta(t) + \delta\big(t-\tfrac{1}{2}\big) + \delta(t-1)\Big)$$

e.5-10

Prove that the Fourier transform of signal $x(t) = \sin(2\pi f_0 t)$ reads:

$$X(f) = j\frac{1}{2}\delta(f+f_0) - j\frac{1}{2}\delta(f-f_0) \tag{5.17}$$

Hint: use the inverse Fourier transform.

e|5.9 Parseval: Fourier transform

e.5-11

Prove that the energy of a signal in time equals the energy of that signal in frequency as defined by Parseval's theorem in (5.19):

$$E = \int_{-\infty}^{\infty} x^2(t)\,\mathrm{d}t = \int_{-\infty}^{\infty} |X(f)|^2\,\mathrm{d}f$$

e|5.c Computer exercises

e.5-12

In Example 5.8, the contribution to the energy of a unit pulse $x(t) = \Pi\left(\frac{t}{\tau}\right)$ from a selection of frequencies was computed using its Fourier transform $X(f) = \tau\,\mathrm{sinc}(\tau f)$, (5.11). The pulse width τ was set to 1 s, causing the zero-crossings of $X(f)$ to occur at $f = m$ Hz $\forall m \in \mathbb{Z}\setminus\{0\}$. It was found that the central lobe of $X(f)$, which lies between the first zero-crossings of $X(f)$ (at -1 and 1 Hz), contains $\approx 90.3\%$ of the total energy of 1 J, a result obtained by numerical integration of $|X(f)|^2$ from -1 to 1 Hz.

In this exercise, we compute the contribution to the total energy of an arbitrary pulse

$$y(t) = A\Pi\left(\frac{t}{\tau}\right)$$

with amplitude A and pulse width τ s, for various frequency bands. The total energy of $y(t)$ can be easily computed in the time domain: it equals $E_y = A^2\tau$, see Figure 2.14 and the discussion on page 27 of the companion book, Theory.

(a) Construct the Fourier transform $Y(f)$ of $y(t)$, for $A = 3$ and $\tau = 2$, and plot the square of this function, $|Y(f)|^2$, against frequency. Make sure that your plot shows a few zero-crossings of $|Y(f)|^2$. Because $|Y(f)|^2$ decreases rapidly for larger $|f|$, plot it also with a logarithmic vertical axis.
Hint: in Python, one can change the vertical axis to log scaling by using the function `matplotlib.pyplot.semilogy` to plot the data.

(b) Compute the contribution of the central lobe of $Y(f)$ to the total energy of the signal E_y, as a fraction of this total energy (denoted by CF below). Note that because $|Y(f)|^2$ is an even function of frequency, you can numerically integrate $|Y(f)|^2$ from $f = 0$ to its first zero-crossing for $f > 0$, at $f = \frac{1}{\tau}$, then multiply by 2. That is:

$$CF = \frac{1}{E_y}\int_{f=-\frac{1}{\tau}}^{f=\frac{1}{\tau}} |Y(f)|^2 \,\mathrm{d}f = \frac{2}{A^2\tau}\int_{f=0}^{f=\frac{1}{\tau}} |Y(f)|^2 \,\mathrm{d}f$$

You can use, e.g., Simpson's rule to perform the numerical integration (in Python: `scipy.integrate.simpson`).

(c) Compute the contribution to the total energy of the signal E_y, as a fraction CF_K, for a range of frequencies that starts at the Kth zero-crossing to the left of $f = 0$ and ends at the Kth zero-crossing to the right of $f = 0$, with $K \in \mathbb{N}^+$.

(d) How large should K be, such that at least 99% of the signal's total energy is included?

(e) And, how large should K be for at least 99.9%?

(f) Repeat your calculations for another pulse amplitude and width. Explain the result.

Solutions

s|5.4 Interpretation, polar form

s.5-1

Use Euler's formula $e^{-j2\pi ft} = \cos(2\pi ft) - j\sin(2\pi ft)$, (C.1), to rewrite the definition of the Fourier transform (5.1):

$$X(f) = \int_{t=-\infty}^{\infty} x(t)e^{-j2\pi ft}\,\mathrm{d}t = \int_{t=-\infty}^{\infty} x(t)\cos(2\pi ft)\,\mathrm{d}t - j\int_{t=-\infty}^{\infty} x(t)\sin(2\pi ft)\,\mathrm{d}t$$

giving the real and imaginary parts of $X(f)$ through the first and second integrals on the right, respectively. Now, it is easy to show that:[1]

$\mathrm{Re}(X(-f)) = \mathrm{Re}(X(f))$, that is, the real part is an even function of f.

$\mathrm{Im}(X(-f)) = -\mathrm{Im}(X(f))$, that is, the imaginary part is an odd function of f.

The *amplitude* of $X(f)$, $|X(f)|$, equals $\sqrt{\mathrm{Re}(X(f))^2 + \mathrm{Im}(X(f))^2}$, (5.6). Hence:

$$\begin{aligned}|X(-f)| &= \sqrt{\mathrm{Re}(X(-f))^2 + \mathrm{Im}(X(-f))^2} = \sqrt{\mathrm{Re}(X(f))^2 + (-\mathrm{Im}(X(f)))^2}\\ &= \sqrt{\mathrm{Re}(X(f))^2 + \mathrm{Im}(X(f))^2} = |X(f)| \qquad \textbf{qed}\end{aligned}$$

The *phase* of $X(f)$, $\angle X(f)$, equals $\arctan\left(\frac{\mathrm{Im}(X(f))}{\mathrm{Re}(X(f))}\right)$, (5.7). Hence:

$$\angle X(-f) = \arctan\left(\frac{\mathrm{Im}(X(-f))}{\mathrm{Re}(X(-f))}\right) = \arctan\left(\frac{-\mathrm{Im}(X(f))}{\mathrm{Re}(X(f))}\right) = -\arctan\left(\frac{\mathrm{Im}(X(f))}{\mathrm{Re}(X(f))}\right) = -\angle X(f) \qquad \textbf{qed}$$

Concluding, the amplitude of $X(f)$ is a real and even function of f, and the phase is a real and odd function of f.

[1] *Hint*: $\cos(2\pi(-f)) = \cos(2\pi f)$ and $\sin(2\pi(-f)) = -\sin(2\pi f)$.

s|5.5 Effects of symmetry

s.5-2

Note that this exercise is similar to Exercise *e*.4-5.

(a) Recall that for an even function: $x(t) = x(-t)$. An example is the cosine function. Use Euler's formula $e^{-j2\pi ft} = \cos(2\pi ft) - j\sin(2\pi ft)$, (C.1), to rewrite the definition of the Fourier transform (5.1):

$$\begin{aligned}X(f) &= \int_{t=-\infty}^{\infty} x(t)e^{-j2\pi ft}\,\mathrm{d}t = \int_{t=-\infty}^{\infty} x(t)\big(\cos(2\pi ft) - j\sin(2\pi ft)\big)\,\mathrm{d}t\\ &= \int_{t=-\infty}^{0} x(t)\big(\cos(2\pi ft) - j\sin(2\pi ft)\big)\,\mathrm{d}t + \int_{t=0}^{\infty} x(t)\big(\cos(2\pi ft) - j\sin(2\pi ft)\big)\,\mathrm{d}t\end{aligned}$$

In the first integral on the right-hand side, we substitute $t = -\sigma$, then $\mathrm{d}t = \mathrm{d}(-\sigma)$; when $t \to -\infty$ then $\sigma \to \infty$, and for $t = 0$, we get $\sigma = 0$:

$$\begin{aligned}X(f) = &\int_{\sigma=\infty}^{0} x(-\sigma)\big(\cos(-2\pi f\sigma) - j\sin(-2\pi f\sigma)\big)\,\mathrm{d}(-\sigma)\\ &+ \int_{t=0}^{\infty} x(t)\big(\cos(2\pi ft) - j\sin(2\pi ft)\big)\,\mathrm{d}t\end{aligned}$$

Since $\cos(-u) = \cos u$ (cosine is even) and $\sin(-u) = -\sin u$ (sine is odd), we get:

$$X(f) = \int_{\sigma=0}^{\infty} x(-\sigma)\big(\cos(2\pi f\sigma) + j\sin(2\pi f\sigma)\big)\,\mathrm{d}\sigma + \int_{t=0}^{\infty} x(t)\big(\cos(2\pi ft) - j\sin(2\pi ft)\big)\,\mathrm{d}t$$

Substituting $\sigma = t$ in the first integral[1] and rearranging yields:

$$X(f) = \int_{t=0}^{\infty} \big(x(t) + x(-t)\big)\cos(2\pi ft)\,\mathrm{d}t - j\int_{t=0}^{\infty} \big(x(t) - x(-t)\big)\sin(2\pi ft)\,\mathrm{d}t \qquad (*)$$

Clearly then, when $x(t)$ is even, i.e., $x(-t) = x(t)$, the second integral in (*) becomes zero, and the result is:

$$X(f) = 2\int_{t=0}^{\infty} x(t)\cos(2\pi f t)\ \mathrm{d}t \qquad \textbf{qed}$$

When $x(t)$ is real (as are all signals in this book) and even, $X(f)$ is a real and even function of f: $X(f) = X(-f)$.
Please note that it is the evenness of the cosine function, *in frequency*, which causes the evenness of $X(f)$. The property does *not* occur because the multiplication of an even time function $x(t)$ and the even cosine function results in an even function of time. This becomes more clear in (b).

(b) From the derivation in (a), we conclude that when $x(t)$ is odd, i.e., $x(-t) = -x(t)$, the first integral in (*) becomes zero, and the result is:

$$X(f) = -2j\int_{t=0}^{\infty} x(t)\sin(2\pi f t)\ \mathrm{d}t \qquad \textbf{qed}$$

When $x(t)$ is real and odd, $X(f)$ is an imaginary and odd function of f: $X(f) = -X(-f)$. Note that it is the oddness of the sine function, *in frequency*, which causes the oddness of $X(f)$. Clearly, the multiplication of an odd time function $x(t)$ and the odd sine function results in an even function of time.

[1]If you do not understand this substitution, please have a look at the footnote at the end of Solution S.3-6.

S.5-3

(I) True, $X(0)$ can be computed using (5.8): $X(0) = \int_{t=-\infty}^{\infty} x(t)\,\mathrm{d}t$, which looking at the shape of signal $x(t)$ equals 0, because as much 'area' appears above the time axis as below it.

(II) False, as half-wave odd symmetry can only be a property of a periodic signal and $x(t)$ is clearly not a periodic signal.

(III) True, as in case of $a = b$, signal $x(t)$ will be odd, i.e., $x(t) = -x(-t)$, and for all odd signals, $X(f)$ will be a purely imaginary function of f.

S.5-4

Statement A. is correct.

For $X(f)$ to be a purely imaginary function of f, $x(t)$ should be (real and) odd. That is clearly not the case here. Hence, statement B. is incorrect.

For $X(f)$ to be a purely real function of f, $x(t)$ should be (real and) even. That is also not the case here, as $x(t)$ would have to be shifted *to the left* to make the signal even. Hence, statement C. is also incorrect.

$X(0)$ can be computed using (5.8): $X(0) = \int_{t=-\infty}^{\infty} x(t)\,\mathrm{d}t$, which looking at the shape of signal $x(t)$ equals $A\tau$. Hence, statement A. is correct, and thus is statement D. incorrect.

s|5.7 Examples

S.5-5

(a) Directly substitute $x_a(t)$ into the Fourier integral (5.1):

$$X_a(f) = \int_{-\infty}^{\infty} x_a(t)e^{-j2\pi ft}\,\mathrm{d}t = \int_{-\infty}^{\infty} te^{-\alpha t}u(t)e^{-j2\pi ft}\,\mathrm{d}t = \int_{0}^{\infty} te^{-(\alpha+j2\pi f)t}\,\mathrm{d}t$$

Using the partial integration rule that states that $\int_a^b \frac{\mathrm{d}f(x)}{\mathrm{d}x} g(x)\,\mathrm{d}x = [f(x)g(x)]_a^b - \int_a^b f(x)\frac{\mathrm{d}g(x)}{\mathrm{d}x}\,\mathrm{d}x$: define $\frac{\mathrm{d}f(t)}{\mathrm{d}t} = e^{-(\alpha+j2\pi f)t}$ and $g(t) = t$, from which it follows that $f(t) = \frac{-1}{\alpha+j2\pi f}e^{-(\alpha+j2\pi f)t}$ and $\frac{\mathrm{d}g(t)}{\mathrm{d}t} = 1$, and we obtain:

$$X_a(f) = \left[\frac{-t}{\alpha+j2\pi f}e^{-(\alpha+j2\pi f)t}\right]_0^{\infty} - \int_0^{\infty}\frac{-1}{\alpha+j2\pi f}e^{-(\alpha+j2\pi f)t}\,\mathrm{d}t$$

The first term on the right-hand side equals zero and integrating the second term yields:

$$X_a(f) = \frac{1}{\alpha+j2\pi f}\left[\frac{-1}{\alpha+j2\pi f}e^{-(\alpha+j2\pi f)t}\right]_0^{\infty} = \frac{1}{(\alpha+j2\pi f)^2}$$

(b) Note that $u(t)u(1-t)$ means that At^2 only needs to be evaluated from $t=0$ to $t=1$ s:

$$X_b(f) = \int_{-\infty}^{\infty} x_b(t)e^{-j2\pi ft}\,\mathrm{d}t = \int_{-\infty}^{\infty} At^2u(t)\,u(1-t)e^{-j2\pi ft}\,\mathrm{d}t = A\int_0^1 t^2e^{-j2\pi ft}\,\mathrm{d}t$$

Integration by parts with $\frac{\mathrm{d}f(t)}{\mathrm{d}t} = e^{-j2\pi ft}$ and $g(t) = t^2$, from which it follows that $f(t) = \frac{-1}{j2\pi f}e^{-j2\pi ft}$ and $\frac{\mathrm{d}g(t)}{\mathrm{d}t} = 2t$, gives:

$$X_b(f) = A\left(\left[\frac{-t^2}{j2\pi f}e^{-j2\pi ft}\right]_0^1 - \frac{-2}{j2\pi f}\int_0^1 te^{-j2\pi ft}\,\mathrm{d}t\right) = A\left(\frac{-1}{j2\pi f}e^{-j2\pi f} + \frac{2}{j2\pi f}\int_0^1 te^{-j2\pi ft}\,\mathrm{d}t\right)$$

The integral in the second term on the right-hand side requires another, but very similar, integration by parts and equals:

$$\int_0^1 te^{-j2\pi ft}\,\mathrm{d}t = \left[\frac{-t}{j2\pi f}e^{-j2\pi ft}\right]_0^1 - \frac{-1}{j2\pi f}\int_0^1 e^{-j2\pi ft}\,\mathrm{d}t = \frac{-1}{j2\pi f}e^{-j2\pi f} + \frac{1}{j2\pi f}\int_0^1 e^{-j2\pi ft}\,\mathrm{d}t$$

$$= \frac{-1}{j2\pi f}e^{-j2\pi f} + \frac{1}{j2\pi f}\left(\frac{-1}{j2\pi f}\right)\left[e^{-j2\pi ft}\right]_0^1 = \frac{-1}{j2\pi f}e^{-j2\pi f} - \frac{1}{(j2\pi f)^2}\left(e^{-j2\pi f}-1\right)$$

Then we obtain:

$$X_b(f) = A\left(\frac{-1}{j2\pi f}e^{-j2\pi f} + \frac{2}{j2\pi f}\left(\frac{-1}{j2\pi f}e^{-j2\pi f} - \frac{1}{(j2\pi f)^2}\left(e^{-j2\pi f}-1\right)\right)\right)$$

$$= \frac{A}{j2\pi f}e^{-j2\pi f}\left(\frac{-1}{j\pi f}-1\right) + \frac{2A}{(j2\pi f)^3}\left(1-e^{-j2\pi f}\right)$$

(c) Just as in (b), $Ae^{-\alpha t}$ only needs to be evaluated from $t=0$ to $t=1$ s:

$$X_c(f) = \int_{-\infty}^{\infty} x_c(t)e^{-j2\pi ft}\,\mathrm{d}t = \int_{-\infty}^{\infty} Ae^{-\alpha t}u(t)\,u(1-t)e^{-j2\pi ft}\,\mathrm{d}t = A\int_0^1 e^{-\alpha t}e^{-j2\pi ft}\,\mathrm{d}t$$

$$= A\int_0^1 e^{-(j2\pi f+\alpha)t}\,\mathrm{d}t = \frac{-A}{j2\pi f+\alpha}\left[e^{-(j2\pi f+\alpha)t}\right]_0^1 = \frac{-A}{j2\pi f+\alpha}\left(e^{-(j2\pi f+\alpha)}-1\right)$$

$$= \frac{A}{j2\pi f+\alpha}\left(1-e^{-(j2\pi f+\alpha)}\right)$$

(d)
$$X_d(f) = \int_{-\infty}^{\infty} x_d(t)e^{-j2\pi ft}\,\mathrm{d}t = \int_{-\infty}^{\infty} e^{-\pi\left(\frac{t}{\tau}\right)^2}e^{-j2\pi ft}\,\mathrm{d}t = \int_{-\infty}^{\infty} e^{-\frac{\pi}{\tau^2}t^2-j2\pi ft}\,\mathrm{d}t$$

Using the given hint, with $a = \frac{\sqrt{\pi}}{\tau}$ and $b = -j2\pi f$. we arrive at:

$$X_d(f) = \frac{\sqrt{\pi}}{a}e^{\frac{b^2}{4a^2}} = \sqrt{\pi}\frac{\tau}{\sqrt{\pi}}e^{\frac{(-j2\pi f)^2}{4}\frac{\tau^2}{\pi}} = \tau e^{\frac{-(2\pi f)^2}{4}\frac{\tau^2}{\pi}} = \tau e^{-\pi(\tau f)^2}$$

S.5-6

We note that $X(f)$ is a real and even function of f, so we expect that when inverse Fourier-transforming, we obtain a real and even function of t. Using (5.2):

$$x(t)=\int_{-\infty}^{\infty}X(f)e^{j2\pi ft}\,\mathrm{d}f=\int_{-\infty}^{0}Ae^{\alpha f}e^{j2\pi ft}\,\mathrm{d}f+\int_{0}^{\infty}Ae^{-\alpha f}e^{j2\pi ft}\,\mathrm{d}f=i_1(t)+i_2(t)$$

Let us start off with $i_2(t)$:

$$i_2(t)=\int_0^{\infty}Ae^{-\alpha f}e^{j2\pi ft}\,\mathrm{d}f=A\int_0^{\infty}e^{(j2\pi t-\alpha)f}\,\mathrm{d}f=\frac{A}{j2\pi t-\alpha}\left[e^{(j2\pi t-\alpha)f}\right]_0^{\infty}$$
$$=\frac{A}{j2\pi t-\alpha}(0-1)=\frac{A}{\alpha-j2\pi t}$$

where in the latter step we used the fact that $e^{(j2\pi t-\alpha)f}=e^{(j2\pi t)f}e^{-\alpha f}$.
For $f\to\infty$, the second exponential becomes 0. The value of the first exponential is never larger than 1, as it is a rotating complex vector with magnitude 1.
Similarly:

$$i_1(t)=\int_{-\infty}^{0}Ae^{\alpha f}e^{j2\pi ft}\,\mathrm{d}f=A\int_{-\infty}^{0}e^{(j2\pi t+\alpha)f}\,\mathrm{d}f=\frac{A}{j2\pi t+\alpha}\left[e^{(j2\pi t+\alpha)f}\right]_{-\infty}^{0}$$
$$=\frac{A}{j2\pi t+\alpha}(1-0)=\frac{A}{j2\pi t+\alpha}$$

Hence:

$$x(t)=\frac{A}{\alpha-j2\pi t}+\frac{A}{j2\pi t+\alpha}=\frac{2A\alpha}{\alpha^2+(2\pi t)^2},$$ which is a real and even function of t.

The exercise ends here.

Furthermore: $x(0)=\frac{2A}{\alpha}$, which equals the value of the integral of $X(f)$ from $-\infty$ to ∞.[1]
Similarly: $X(0)=A$, which equals the value of the integral of $x(t)$ from $-\infty$ to ∞.[2]

[1] $\int_{-\infty}^{\infty}Ae^{-\alpha|f|}\,\mathrm{d}f=2A\int_0^{\infty}e^{-\alpha f}\,\mathrm{d}f=\frac{2A}{-\alpha}\left[e^{-\alpha f}\right]_0^{\infty}=\frac{2A}{-\alpha}(0-1)=\frac{2A}{\alpha}$.

[2] $\int_{-\infty}^{\infty}x(t)\,\mathrm{d}t=2\int_0^{\infty}\frac{2A\alpha}{\alpha^2+(2\pi t)^2}\,\mathrm{d}t$. From calculus we know that $\int\frac{1}{a^2+b^2u^2}\,\mathrm{d}u=\frac{1}{ab}\arctan\left(\frac{b}{a}u\right)$. We use this result here, with $a=\alpha$ and $b=2\pi$, to obtain: $2\int_0^{\infty}\frac{2A\alpha}{\alpha^2+(2\pi t)^2}\,\mathrm{d}t=4A\alpha\frac{1}{\alpha 2\pi}\left[\arctan\left(\frac{2\pi}{\alpha}t\right)\right]_0^{\infty}=\frac{2A}{\pi}\left(\frac{\pi}{2}-0\right)=A$.

S.5-7

Let $y(t)$ be the aperiodic signal corresponding to just one period T_0 of pulse train signal $x(t)$.
This aperiodic signal can be defined as: $y(t)=A\Pi\left(\frac{t}{\tau}\right)$

Fourier-transforming $y(t)$: $Y(f)=\mathcal{F}\left\{A\Pi\left(\frac{t}{\tau}\right)\right\}=A\tau\,\mathrm{sinc}(\tau f)$ (see (5.11) in Example 5.1)

Using $X_k=\frac{1}{T_0}X(f=kf_0)$ from Section 5.1:

$$X_k=\frac{1}{T_0}A\tau\,\mathrm{sinc}(kf_0\tau)=\frac{A\tau}{T_0}\mathrm{sinc}\left(k\frac{\tau}{T_0}\right)\quad\text{with}\quad k\in\mathbb{Z}$$ **qed**

S.5-8

Pulse function $z(t)$ has amplitude $A=2$ and width $\tau=\frac{T_0}{2}$ s centered at $t=0$ s:

$$z(t)=2\Pi\left(\frac{t}{\frac{T_0}{2}}\right)$$

Using $\Pi\left(\frac{t}{\tau}\right)\overset{\mathcal{F}}{\leftrightarrow}\tau\,\mathrm{sinc}(\tau f)$, (5.11):

$$Z(f)=2\frac{T_0}{2}\mathrm{sinc}\left(\frac{T_0}{2}f\right)=T_0\,\mathrm{sinc}\left(\frac{T_0}{2}f\right)$$

The difference between $z(t)$ and a single period of $y(t)$ centered at $t=0$ s is that $z(t)$ is shifted upwards by 1:

$$z(t)=y_{-T_0/2,T_0/2}(t)+1$$

When considering a periodic signal, adding a non-zero constant to it affects only one Fourier series coefficient. Only the coefficient with index $k=0$, which represents the signal's average, changes; all other coefficients remain the same.

Using $X_k = \frac{1}{T_0} X(f = kf_0)$ from Section 5.1, we obtain the complex exponential Fourier series coefficients of a periodic signal $z_{\text{per}}(t)$ consisting of periodic repetitions (with period T_0) of pulse function $z(t)$ (taken from $t = -\frac{T_0}{2}$ to $t = \frac{T_0}{2}$):

$$Z_{\text{per},k} = \frac{1}{T_0} Z(f = kf_0) = \frac{1}{T_0} T_0 \operatorname{sinc}\left(\frac{T_0}{2} kf_0\right) = \operatorname{sinc}\left(\frac{T_0}{2} k \frac{1}{T_0}\right) = \operatorname{sinc}\left(\frac{1}{2} k\right)$$

Using $\operatorname{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$, (B.3):

$$Z_{\text{per},k} = \frac{\sin\left(k\frac{\pi}{2}\right)}{k\frac{\pi}{2}} = \frac{2\sin\left(k\frac{\pi}{2}\right)}{k\pi}$$

$Z_{\text{per},k}$ equals zero when $\sin\left(k\frac{\pi}{2}\right)$ equals zero.

Hence: $Z_{\text{per},k} = 0$ for $k\frac{1}{2} = \pm 1, \pm 2, \pm 3, \pm 4, \ldots$, and thus for $k = \pm 2, \pm 4, \pm 6, \pm 8, \ldots$

i.e., $Z_{\text{per},k} = 0$ for k even

(this does *not* include $k = 0$, as the average of $z_{\text{per}}(t)$ does not equal zero.)

When considering k odd, we obtain for negative k: $\sin\left(k\frac{\pi}{2}\right) = -\sin\left(|k|\frac{\pi}{2}\right)$ and $k\pi = -|k|\pi$

Hence, for both positive and negative k odd, we obtain:

$$Z_{\text{per},k} = \operatorname{sinc}\left(\frac{1}{2}k\right) = \frac{2\sin\left(|k|\frac{\pi}{2}\right)}{|k|\pi}$$

For $k = 1,\ 5,\ 9, \ldots$: $\sin\left(k\frac{\pi}{2}\right) = 1$ For $k = 3,\ 7,\ 11, \ldots$: $\sin\left(k\frac{\pi}{2}\right) = -1$

Using the above, we obtain:

$$Z_{\text{per},k} = \begin{cases} +\frac{2}{|k|\pi} & \text{for } k = \pm 1, \pm 5, \pm 9, \ldots \\ -\frac{2}{|k|\pi} & \text{for } k = \pm 3, \pm 7, \pm 11, \ldots \end{cases}$$

The only difference between $z_{\text{per}}(t)$ and $y(t)$ is that $z_{\text{per}}(t)$ is shifted upwards by 1 compared to $y(t)$. As a result, all coefficients Y_k are identical to coefficients $Z_{\text{per},k}$, except for Y_0. Coefficient Y_0 represents the average of signal $y(t)$, and from the illustration of $y(t)$, it is clear that the signal's average is zero.

Concluding: $$Y_k = \begin{cases} +\frac{2}{|k|\pi} & \text{for } k = \pm 1, \pm 5, \pm 9, \ldots \\ -\frac{2}{|k|\pi} & \text{for } k = \pm 3, \pm 7, \pm 11, \ldots \\ 0 & \text{for } k \text{ even (including } k = 0) \end{cases}$$

s | 5.8 Fourier transform in-the-limit

S.5-9

This signal is real and even, so its Fourier transform will also be real and even. We compute the Fourier transform through direct evaluation using (5.1):

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft}\, \mathrm{d}t$$

$$= \int_{-\infty}^{\infty} \left(\frac{1}{4}\left(\delta(t+1) + \delta\left(t + \frac{1}{2}\right) + \delta(t) + \delta\left(t - \frac{1}{2}\right) + \delta(t-1)\right)\right) e^{-j2\pi ft}\, \mathrm{d}t$$

Using the sifting property (B.12) of the Dirac delta function, we obtain:

$$X(f) = \frac{1}{4}\left(e^{j2\pi f} + e^{j\pi f} + 1 + e^{-j\pi f} + e^{-j2\pi f}\right)$$

Using $\frac{1}{2}\left(e^{-j\theta} + e^{j\theta}\right) = \cos\theta$, (C.2):

$X(f) = \frac{1}{4} + \frac{1}{2}\cos(2\pi f) + \frac{1}{2}\cos(\pi f)$, which is a real and even function of frequency f.

s.5-10 Starting from the given expression for $X(f)$ and using the inverse Fourier transform (5.2):

$$x(t) = \int_{-\infty}^{\infty} X(f)e^{j2\pi ft}\,\mathrm{d}f = \int_{-\infty}^{\infty}\left(j\tfrac{1}{2}\delta(f+f_0) - j\tfrac{1}{2}\delta(f-f_0)\right)e^{j2\pi ft}\,\mathrm{d}f$$

Using the sifting property (B.12) of the Dirac delta function and $j\frac{1}{2}\left(e^{-j\theta} - e^{j\theta}\right) = \sin\theta$, (C.3):

$$x(t) = j\tfrac{1}{2}e^{-j2\pi f_0 t} - j\tfrac{1}{2}e^{j2\pi f_0 t} = \sin(2\pi f_0 t)$$ **qed**

s | 5.9 Parseval: Fourier transform

s.5-11 To prove: $\displaystyle\int_{-\infty}^{\infty} x^2(t)\,\mathrm{d}t = \int_{-\infty}^{\infty} |X(f)|^2\,\mathrm{d}f$

Recall the Fourier transform $\displaystyle X(f) = \int_{t=-\infty}^{\infty} x(t)e^{-j2\pi ft}\,\mathrm{d}t$ (5.1)

and the inverse Fourier transform $\displaystyle x(t) = \int_{f=-\infty}^{\infty} X(f)e^{j2\pi ft}\,\mathrm{d}f$ (5.2)

Then: $\displaystyle\int_{f=-\infty}^{\infty} |X(f)|^2\,\mathrm{d}f = \int_{f=-\infty}^{\infty} X(f)X(-f)\,\mathrm{d}f$

Insert (5.1) into the above equation:

$$\int_{f=-\infty}^{\infty} |X(f)|^2\,\mathrm{d}f = \int_{f=-\infty}^{\infty}\left(\int_{t=-\infty}^{\infty} x(t)e^{-j2\pi ft}\,\mathrm{d}t\right)X(-f)\,\mathrm{d}f$$

Then change the order of integration:

$$\int_{f=-\infty}^{\infty} |X(f)|^2\,\mathrm{d}f = \int_{t=-\infty}^{\infty}\int_{f=-\infty}^{\infty} x(t)e^{-j2\pi ft}X(-f)\,\mathrm{d}f\,\mathrm{d}t$$

Take all elements that are *not* a function of f out of the 'inner' integral (here, just $x(t)$):

$$\int_{f=-\infty}^{\infty} |X(f)|^2\,\mathrm{d}f = \int_{t=-\infty}^{\infty} x(t)\int_{f=-\infty}^{\infty} e^{-j2\pi ft}X(-f)\,\mathrm{d}f\,\mathrm{d}t$$

On the right-hand side, in the integral with f, we substitute $f = -\nu$, then $\mathrm{d}f = \mathrm{d}(-\nu) = -\,\mathrm{d}\nu$; when $f \to -\infty$ then $\nu \to \infty$, and when $f \to \infty$ then $\nu \to -\infty$:

$$\int_{f=-\infty}^{\infty} |X(f)|^2\,\mathrm{d}f = -\int_{t=-\infty}^{\infty} x(t)\int_{\nu=\infty}^{-\infty} e^{j2\pi\nu t}X(\nu)\,\mathrm{d}\nu\,\mathrm{d}t$$

We can reverse the interval limits, which causes an additional minus sign:

$$\int_{f=-\infty}^{\infty} |X(f)|^2\,\mathrm{d}f = \int_{t=-\infty}^{\infty} x(t)\int_{\nu=-\infty}^{\infty} X(\nu)e^{j2\pi\nu t}\,\mathrm{d}\nu\,\mathrm{d}t$$

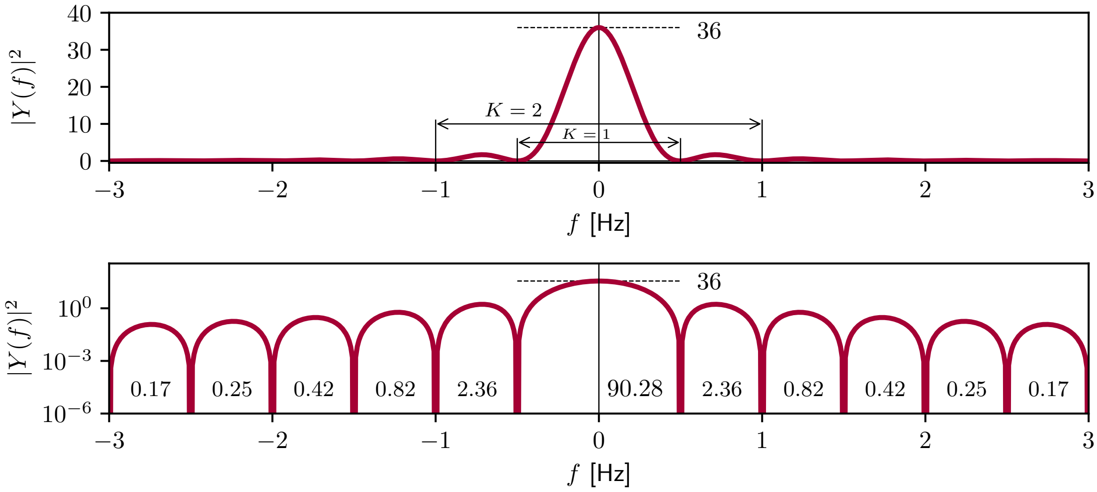
We recognize the ν-integral to be the inverse Fourier transform. Hence, using (5.2) we obtain:

$$\int_{f=-\infty}^{\infty} |X(f)|^2\,\mathrm{d}f = \int_{t=-\infty}^{\infty} x(t)x(t)\,\mathrm{d}t = \int_{t=-\infty}^{\infty} x^2(t)\,\mathrm{d}t$$ **qed**

S | 5.c Computer exercises

S.5-12

(a) The figure below shows $|Y(f)|^2$ on a linear scale at top; we see that the lobes rapidly become smaller, and so does the contribution of these lobes to the total signal energy. The bottom graph uses a logarithmic vertical axis to present $|Y(f)|^2$.



(b) The central lobe contains $\approx 90.28\%$ of the total signal energy E_y, exactly the same as in Example 5.8 (with $A = 1$ and $\tau = 1$; here we used $A = 3$ and $\tau = 2$ in our calculations).

(c) The numbers inside the lobes in the figure above represent the percentage contributions (= fractions times 100%) of the respective lobes to the total signal energy E_y. For example, for $K = 2$, the first side lobes on either side of $f = 0$ together contain $\approx 4.72\%$ of the signal energy (2 times 2.36% in the graph). Therefore, for $K = 2$ the contribution to the energy equals $90.28 + 4.72 = 95\%$.

(d) For $K \geq 11$ (not shown), we pass the 99% point; that is, when integrating from $f = -\frac{K}{\tau} = -\frac{11}{2} = -5.5$ Hz to 5.5 Hz, more than 99% of the signal's energy is included.

(e) For $K \geq 101$ (not shown), we pass the 99.9% point: when integrating from $f = -\frac{101}{2} = -50.5$ Hz to 50.5 Hz, we include more than 99.9% of the signal energy.

(f) The contribution fractions of the lobes are the same for all pulses, no matter amplitude A and width τ. That is because we divide by the total energy $E_y = A^2\tau$:

$$CF_K = \frac{1}{A^2\tau}\int_{f=-\frac{K}{\tau}}^{\frac{K}{\tau}} A^2\tau^2\,\mathrm{sinc}^2(\tau f)\,\mathrm{d}f = \tau\int_{-\frac{K}{\tau}}^{\frac{K}{\tau}} \frac{\sin^2(\pi\tau f)}{(\pi\tau f)^2}\,\mathrm{d}f$$

$$= \tau\frac{1}{\pi\tau}\int_{-K\pi}^{K\pi}\frac{\sin^2 u}{u^2}\,\mathrm{d}u = \frac{1}{\pi}\int_{-K\pi}^{K\pi}\frac{\sin^2 u}{u^2}\,\mathrm{d}u = \frac{2}{\pi}\int_{0}^{K\pi}\frac{\sin^2 u}{u^2}\,\mathrm{d}u$$

where we substituted $\pi\tau f = u$ on the second line. We see that the expression for CF_K does not depend on A or τ, it only depends on K, with $K \in \mathbb{N}^+$. Using (A.15), when $K \to \infty$, we obtain $CF_{K\to\infty} = \frac{2}{\pi}\frac{\pi}{2} = 1$, as expected.

Basic Python code:

```
import numpy as np
from scipy.integrate import simpson
from matplotlib import pyplot as plt

burgundy = [165/255,0/255,52/255]# burgundy color
```

```
# pulse signal y(t) with amplitude A and width tau s
A   = 3                          # pulse amplitude [-]
tau = 2                          # pulse width [s]
totalenergy = A*A*tau            # signal energy
zcross = 1/tau                   # base factor for zero-crossings

# create Fourier transform Y(f) of pulse signal y(t)
f   = np.linspace(-3, 3, 601)    # frequency steps 0.01 Hz from -3 to 3 Hz
Yf  = A*tau*np.sinc(tau*f)       # Fourier transform Y(f)
Yf2 = Yf*Yf                      # Y(f)^2
Yf2max = A*A*tau*tau             # maximum value of Y(f)^2

# plot |Y(f)|^2 as function of frequency
plt.figure()
plt.subplot(3,1,1)               # linear vertical axis
plt.plot(f, Yf2, color=burgundy, linewidth=2)
plt.xlabel('$f$ [Hz]'); plt.ylabel('$|Y(f)|^2$')
plt.subplot(3,1,2)               # logarithmic vertical axis
plt.semilogy(f, Yf2, color=burgundy, linewidth=2)
plt.xlabel('$f$ [Hz]'); plt.ylabel('$|Y(f)|^2$')

# compute energy from -K/tau to +K/tau with K integer (1,K)
K=10
penergy = np.zeros(K+1,); cpenergy = np.zeros(K+1,); oldenergy = 0;
for kk in range(1,K+1):
    # define frequency axis in Hz with sufficient resolution
    fmax = kk/tau; fstep = fmax/(kk*10000)
    f = np.arange(0, fmax+fstep, fstep)
    # define Fourier transform Y(f) of pulse y(t) and Y(f)^2
    Yff = A*tau*np.sinc(tau*f); Yff2 = Yff*Yff
    # integrate |Y(f)|^2 over the frequency range [0, fmax] using Simpson,
    # then multiply by 2 (possible as |Y(f)|^2 is an even function of f)
    energy = 2 * simpson(Yff2, f)
    # compute delta-energy and compose energy percentage vector
    denergy = energy - oldenergy; oldenergy = energy
    penergy[kk,]  = 100*(denergy/totalenergy)
    cpenergy[kk,] = cpenergy[kk-1] + penergy[kk]
    # show results
    print(f"{kk:2d} {energy:f} {denergy:f} {penergy[kk]:6.3f} {cpenergy[kk
       ]:6.3f}")

# show percentage contributions of lobes in plot
plt.text( 0.25, 0.00001, f'{"%5.2f"%(penergy[1])}',   ha='center', size=9)
plt.text(-0.75, 0.00001, f'{"%5.2f"%(penergy[2]/2)}', ha='center', size=9)
plt.text( 0.75, 0.00001, f'{"%5.2f"%(penergy[2]/2)}', ha='center', size=9)
```

6

Fourier transform theorems

Exercises

e | 6.2 Time shift

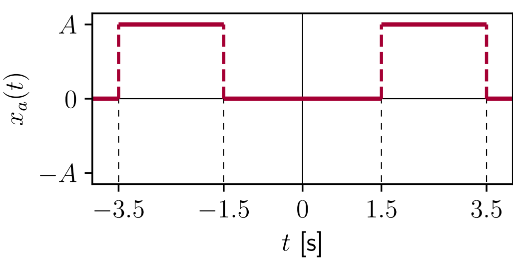
e.6-1

For each of the four signals shown below:

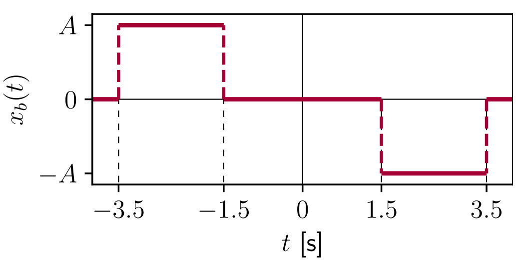
(1) Use the linearity and/or time shift theorem(s) together with the Fourier transform pair $\Pi\left(\frac{t}{\tau}\right) \overset{\mathcal{F}}{\leftrightarrow} \tau\,\mathrm{sinc}(\tau f)$, (5.11), to compute the signal's Fourier transform.

(2) Using symmetry, show that the signal's Fourier transform is real and even, imaginary-valued and odd, or neither.

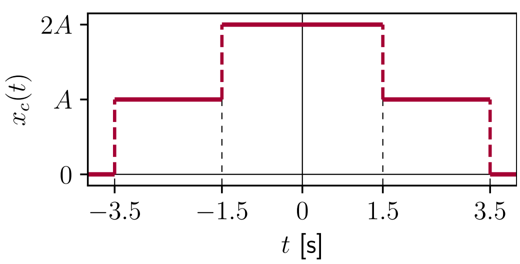
(a)



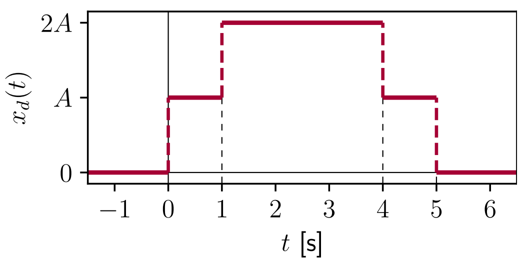
(b)



(c)



(d)

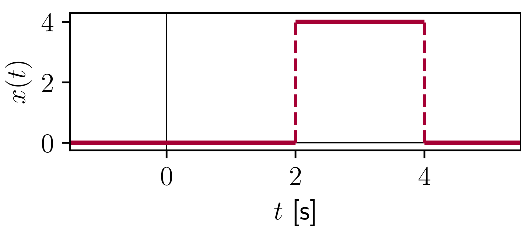


e.6-2

Consider signal $x(t)$ shown on the right. Some operation (for you to determine) is performed on $x(t)$, resulting in signal $y(t)$.
The Fourier transform of $y(t)$, $Y(f)$, equals:

$$Y(f) = 16\,\mathrm{sinc}(2f)\,e^{j6\pi f}$$

Express signal $y(t)$ as a function of signal $x(t)$.



e.6-3 Compute Fourier transform $X(f)$ of the signal

$$x(t) = 2\Pi(-2t-5) + e^{-j10\pi t} + e^{j10\pi t}.$$

e.6-4 Consider the signal

$$x(t) = 2\,\mathrm{sinc}(2t).$$

Define new signal

$$y(t) = x(t+t_0)$$

for an arbitrary time shift t_0 s.

(a) Compute the Fourier transform of $y(t)$.

(b) Sketch $y(t)$, $|Y(f)|$ and $\angle Y(f)$ for three values of t_0:
(1) $t_0 = 0$ s (2) $t_0 = 0.25$ s (3) $t_0 = -0.5$ s

(c) Sketch $\mathrm{Re}(Y(f))$ and $\mathrm{Im}(Y(f))$ for the same three values of t_0: 0, 0.25 and -0.5 s.

(d) Sketch $\angle Y(f)$ for $t_0 = 0.75$ s. Explain the result.

e|6.3 Time scale change

e.6-5 A sensor records a (continuous-time) signal which has an amplitude spectrum that is approximately rectangular, extending from 500 Hz to 2,700 Hz. The signal is stored on a medium that preserves its waveform, but allows the playback speed to be varied. During analysis, the signal is replayed at half the original speed.

Sketch the amplitude spectra of the original (recorded) signal and the replayed signal.

e.6-6 Consider sinusoidal signal

$$x(t) = A\cos(2\pi f_0 t).$$

Define new signal

$$y(t) = x(at),$$

for an arbitrary real number $a > 0$.

(a) Compute the Fourier transform of $y(t)$.

(b) Sketch $x(t)$ and $y(t)$ for $A = 1.5$, $f_0 = 2$ Hz and $a = 3$.

(c) Compute the average power of $x(t)$ and $y(t)$. Explain the result.

e|6.4 Time reversal

e.6-7 Consider the three signals defined below. For each of these signals:

(1) Compute its Fourier transform (for $\alpha > 0$, and with $u(t)$ the unit step function, if applicable).
Hint: start from Fourier transform $Z(f)$ of the function $z(t) = e^{-\alpha t}u(t)$ from Example 5.3, and apply the appropriate Fourier transform theorem(s).

(2) Compute the real and imaginary parts of the Fourier transform.

(3) Compute the amplitude and phase of the Fourier transform.

(a) $a(t) = e^{\alpha t}u(-t)$ (b) $b(t) = e^{-\alpha|t|}$ (c) $c(t) = e^{-\alpha t}u(t) - e^{\alpha t}u(-t)$

e|6.5 Duality

e.6-8

Compute Fourier transform $Y(f)$ of signal $y(t)$, given the following combinations of signals:

(a) $x(t) = 4\,\mathrm{sinc}(5t)$ and $y(t) = \frac{1}{2}x(2t)$

(b) $x(t) = 3\,\mathrm{sinc}(6t)$ and $y(t) = 4x\left(-\frac{1}{2}t\right)$

e.6-9

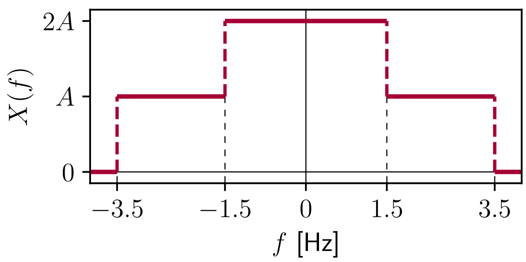
Use the duality theorem to obtain time signal $x(t)$ of which the Fourier transform is

$$X(f) = Ae^{-\alpha|f|} \qquad \text{for } \alpha > 0.$$

Hint: first do Exercise e.6-7(b).

e.6-10

Use the duality theorem to obtain time signal $x(t)$ of which the Fourier transform $X(f)$ is shown here on the right.
Hint: first do Exercise e.6-1(c).



e.6-11

Two voltage signals are given by: $x(t) = 1 + 16\,\mathrm{sinc}(4t)$

$y(t) = 6\cos(30\pi t) + 2\sin(10\pi t)$

A third voltage signal is defined by: $z(t) = x(t) + y(t)$

Determine for each of the three signals whether it is an energy signal or a power signal, and compute its energy (if it is an energy signal) or average power (if it is a power signal).

e|6.7 Modulation

e.6-12

The four signals from Exercise e.6-1 are each multiplied by the signal

$$p(t) = \cos(12\pi t).$$

Compute the Fourier transforms of the resulting signals.

e.6-13

Consider voltage signal

$$x(t) = 10\,\mathrm{sinc}(5t)\cos^2(20\pi t).$$

(a) Is this an energy or a power signal? Explain your answer.

(b) Compute its energy (if it is an energy signal) or average power (if it is a power signal).

e.6-14

Two voltage signals are given by: $x(t) = 8\,\mathrm{sinc}(8t)$

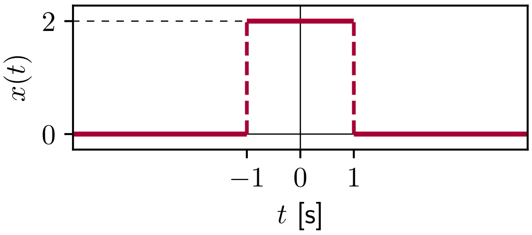
$y(t) = 4\,\mathrm{sinc}(2t)\cos(8\pi t)$

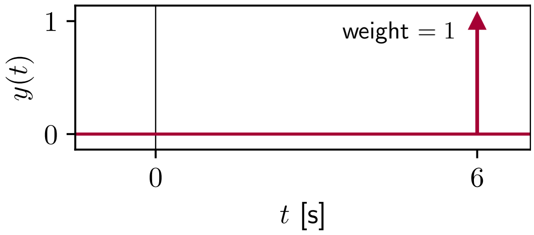
A third voltage signal is defined by: $z(t) = x(t) + y(t)$

All three signals are energy signals. Compute their energies.

e|6.8 Convolution

e.6-15 Consider signals $x(t)$, a pulse function, and $y(t)$, a Dirac delta function with a weight equal to 1, shown below.





A third signal, $z(t)$, is defined as the convolution of $x(t)$ and $y(t)$: $z(t) = x(t) * y(t)$.

Compute the Fourier transform of $z(t)$, $Z(f)$.

e|6.9 Multiplication

e.6-16 A damped cosine, as a function of time t, is given by

$$y(t) = e^{-\alpha t}\cos(2\pi f_0 t) \qquad \text{for } t \geq 0 \text{ and } \alpha > 0.$$

Compute the Fourier transform of $y(t)$, $Y(f)$.

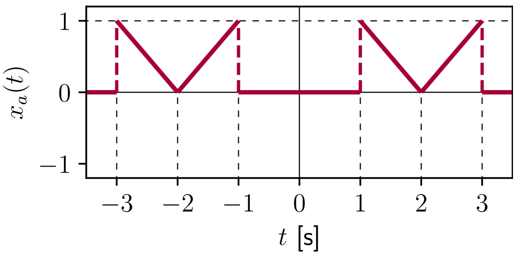
e|6.10 Differentiation

e.6-17 For the two signals shown below:

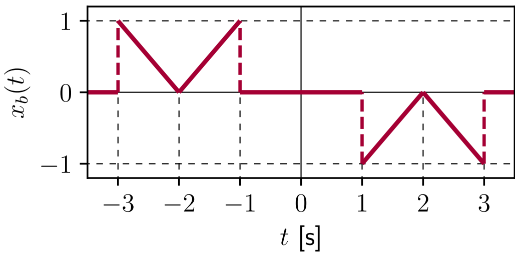
(1) Determine whether the Fourier transform of the signal is real *and* even, or imaginary *and* odd.

(2) Use the differentiation and time shift theorems to compute the signal's Fourier transform.

(a)



(b)



e.6-18 This exercise illustrates a subtle, but important limitation of using the differentiation Fourier transform theorem.

Consider signals $y_a(t)$ and $y_b(t)$ shown below.

(a) Use the differentiation and time shift theorems to compute the Fourier transform of $y_a(t)$.

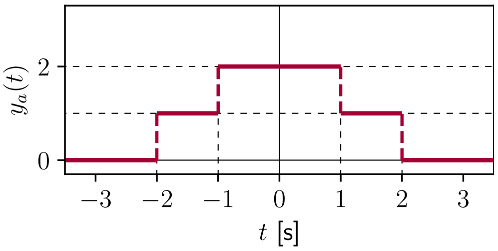
(b) Compute the Fourier transform of $y_a(t)$ again, but this time using the Fourier transform pair $\Pi\left(\frac{t}{\tau}\right) \overset{\mathcal{F}}{\leftrightarrow} \tau\,\mathrm{sinc}(\tau f)$, (5.11). Compare the resulting expression for $Y_a(f)$ with the one obtained in (a).

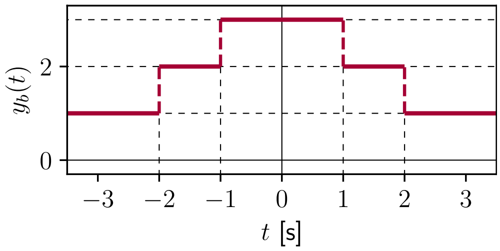
(c) Express signal $y_b(t)$ in terms of $y_a(t)$, and use Fourier transform $Y_a(f)$ obtained in (a) to compute $Y_b(f)$.

(d) Let $b(t)$ denote the derivative of $y_b(t)$, i.e., $b(t) = \frac{\mathrm{d}}{\mathrm{d}t} y_b(t)$. Use $Y_b(f)$ obtained in (c) to compute the Fourier transform of $b(t)$.

(e) Compare the Fourier transform of $b(t)$ obtained in (d) with the Fourier transform of a signal $a(t)$ defined as the derivative of $y_a(t)$, i.e., $a(t) = \frac{\mathrm{d}}{\mathrm{d}t} y_a(t)$.

(f) Can $Y_b(f)$ be uniquely recovered from $B(f)$? Justify your answer.





e|6.c Computer exercises

e.6-19

Consider the sinusoid pulse signal given by

$$x(t) = A\Pi\left(\frac{t}{\tau}\right)\cos(2\pi f_0 t),$$

where A denotes the signal amplitude, and the cosine is multiplied by a square pulse of duration τ.

(a) Plot signal $x(t)$ in the time domain.

(b) Compute the Fourier transform $X(f)$ of this signal.

(c) Plot the amplitude spectrum $|X(f)|$ of this signal.

e.6-20

Consider the three signals from Exercise *e*.6-7 on page 96 as well as their Fourier transforms obtained in that exercise.

(a) Plot time signal $a(t)$ as well as its double-sided amplitude and phase spectra. Compare them with those of signal $z(t)$ shown in Example 5.3. Choose a constant value for α, e.g., 1 or 3, to be able to make a direct comparison with the plots of signal $z(t)$.

(b) Plot time signals $b(t)$ and $c(t)$ as well as their double-sided amplitude and phase spectra, and compare them with one another. Choose a constant value for α, e.g., 1 or 3.

e.6-21

In this exercise, you are required to write a script that allows the 'in-depth' analysis of Exercise *e*.6-16 as will be discussed on page 109.

(a) Plot time signal $y(t)$ (for $0 \leq t \leq 5$ s) and the amplitude spectrum $|Y(f)|$ (for $-5 \leq f \leq 5$ Hz) of the damped cosine signal introduced in Exercise *e*.6-16. Use $\alpha = 1$ and $f_0 = 2$ Hz. How does the Fourier transform of a damped cosine differ from the Fourier transform of a pure (infinitely lasting) cosine with the same frequency $f_0 = 2$ Hz?

(b) Extend your script written for (a) such that in your plots you also show the results for $\alpha = 0.5$ and $\alpha = 2$. Explain what happens when α decreases. What would the response $y(t)$ and its Fourier transform $Y(f)$ look like when $\alpha \downarrow 0$?

Solutions

S | 6.2 Time shift

S.6-1

(a) $x_a(t) = A\,\Pi\left(\frac{t+2.5}{2}\right) + A\,\Pi\left(\frac{t-2.5}{2}\right)$, which is an even function.

We know that $\mathcal{F}\left\{\Pi\left(\frac{t}{\tau}\right)\right\} = \tau\,\mathrm{sinc}(\tau f)$, (5.11), and we can use the time shift theorem (6.2) to compute that $\mathcal{F}\left\{\Pi\left(\frac{t+t_0}{\tau}\right)\right\} = \tau\,\mathrm{sinc}(\tau f)\,e^{j2\pi f t_0}$. Hence:

$$X_a(f) = \mathcal{F}\left\{A\Pi\left(\frac{t+2.5}{2}\right)\right\} + \mathcal{F}\left\{A\Pi\left(\frac{t-2.5}{2}\right)\right\} = 2A\,\mathrm{sinc}(2f)\left(e^{j5\pi f} + e^{-j5\pi f}\right)$$

Using $\frac{1}{2}\left(e^{-j\theta} + e^{j\theta}\right) = \cos\theta$, (C.2):

$X_a(f) = 4A\,\mathrm{sinc}(2f)\cos(5\pi f)$, which is a real and even function of frequency.

(b) $x_b(t) = A\,\Pi\left(\frac{t+2.5}{2}\right) - A\,\Pi\left(\frac{t-2.5}{2}\right)$, which is an odd function.

Following a similar approach to that in (a), but using $j\frac{1}{2}\left(e^{-j\theta} - e^{j\theta}\right) = \sin\theta$, (C.3):

$X_b(f) = 2A\,\mathrm{sinc}(2f)\left(e^{j5\pi f} - e^{-j5\pi f}\right) = 4Aj\,\mathrm{sinc}(2f)\sin(5\pi f)$,

which is an imaginary-valued and odd function of frequency.

(c) $x_c(t) = A\,\Pi\left(\frac{t}{7}\right) + A\,\Pi\left(\frac{t}{3}\right)$, which is an even function.

$X_c(f) = 7A\,\mathrm{sinc}(7f) + 3A\,\mathrm{sinc}(3f)$, which is a real and even function of frequency.[1]

(d) $x_d(t) = A\,\Pi\left(\frac{t-2.5}{5}\right) + A\,\Pi\left(\frac{t-2.5}{3}\right)$ This function has no symmetry properties.

$$X_d(f) = 5A\,\mathrm{sinc}(5f)\,e^{-j5\pi f} + 3A\,\mathrm{sinc}(3f)\,e^{-j5\pi f} = Ae^{-j5\pi f}\left(3\,\mathrm{sinc}(3f) + 5\,\mathrm{sinc}(5f)\right)$$

Clearly, $X_d(f)$ is a complex-valued function. Whereas the two components between parentheses, $3\,\mathrm{sinc}(3f)$ and $5\,\mathrm{sinc}(5f)$ are both real and even functions of frequency, their sum is multiplied by $Ae^{-j5\pi f}$, which is a complex number. Naturally, $X_d(-f) = X_d^*(f)$, but that does not imply any special symmetry.

[1]Another, more cumbersome approach would be to start with defining $x_c(t) = A\Pi\left(\frac{t+2.5}{2}\right) + 2A\Pi\left(\frac{t}{3}\right) + A\Pi\left(\frac{t-2.5}{2}\right)$, and Fourier-transforming, which yields: $X_c(f) = 4A\,\mathrm{sinc}(2f)\cos(5\pi f) + 6A\,\mathrm{sinc}(3f)$, which after some rewriting using (B.3) and (A.10) results in the same answer. However, the approach stated above requires less effort.

S.6-2

From the illustration of signal $x(t)$: $x(t) = 4\Pi\left(\frac{t-3}{2}\right)$

From $\Pi\left(\frac{t}{\tau}\right) \overset{\mathcal{F}}{\leftrightarrow} \tau\,\mathrm{sinc}(\tau f)$, (5.11), and the time shift theorem (6.2), we know that:
$\mathcal{F}\left\{\Pi\left(\frac{t-t_0}{\tau}\right)\right\} = \tau\,\mathrm{sinc}(\tau f)\,e^{-j2\pi f t_0}$. From here on, one can follow two approaches:

(1) Determine $y(t)$ from $Y(f)$ and compare $y(t)$ to $x(t)$ directly:

Given $Y(f) = 16\,\mathrm{sinc}(2f)\,e^{j6\pi f}$ and using the information above: $y(t) = 8\Pi\left(\frac{t+3}{2}\right)$

Compare $y(t)$ to $x(t)$ directly to find out that: $y(t) = 2x(t+6)$

(2) Determine $X(f)$, compare $X(f)$ to $Y(f)$ and then deduce $y(t)$ as a function of $x(t)$:

$$X(f) = \mathcal{F}\left\{4\Pi\left(\frac{t-3}{2}\right)\right\} = 4 \cdot 2\,\mathrm{sinc}(2f)\,e^{-j2\pi f 3} = 8\,\mathrm{sinc}(2f)\,e^{-j6\pi f}$$

Given $Y(f) = 16\,\mathrm{sinc}(2f)\,e^{j6\pi f}$: $Y(f) = 2X(f)e^{j12\pi f}$ Finally: $y(t) = 2x(t+6)$

Hence, the expression for signal $y(t)$ is:

$$y(t) = 2x(t+6)$$

S.6-3

Using $\frac{1}{2}(e^{-j\theta}+e^{j\theta}) = \cos\theta$, (C.2), to rewrite $x(t) = 2\Pi(-2t-5) + e^{-j10\pi t} + e^{j10\pi t}$:

$$x(t) = 2\Pi\big(-2(t+2.5)\big) + 2\cos(10\pi t) = 2\Pi\Big(\frac{t+2.5}{0.5}\Big) + 2\cos(10\pi t),$$

where we used the fact that the unit pulse function is an even function, i.e., $x(-t) = x(t)$ (mirroring about $t = 0$ s has no effect).

From $\Pi\big(\frac{t}{\tau}\big) \overset{\mathcal{F}}{\leftrightarrow} \tau\,\mathrm{sinc}(\tau f)$, (5.11), and the time shift theorem (6.2), we know that: $\mathcal{F}\Big\{\Pi\big(\frac{t+t_0}{\tau}\big)\Big\} = \tau\,\mathrm{sinc}(\tau f)\,e^{j2\pi f t_0}$.

Combining the above with $\mathcal{F}\{\cos(2\pi f_0 t)\} = \frac{1}{2}\big(\delta(f+f_0)+\delta(f-f_0)\big)$, (5.16), we obtain:

$$X(f) = \mathrm{sinc}\Big(\frac{f}{2}\Big)\,e^{j5\pi f} + \delta(f+5) + \delta(f-5)$$

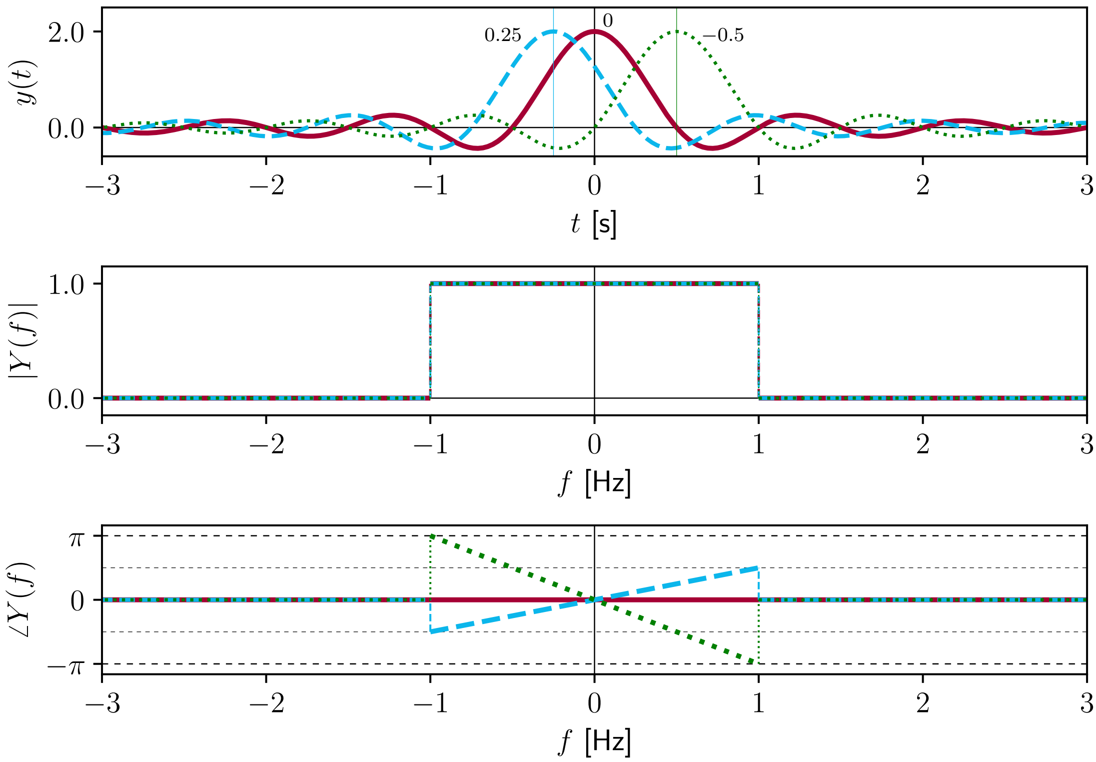
S.6-4

(a) The Fourier transform of $x(t) = 2\,\mathrm{sinc}\,(2t)$ equals $X(f) = \Pi\big(\frac{f}{2}\big)$, (6.13). The real, even signal has a real, even Fourier transform. Using the time shift theorem (6.2), we obtain:

$$Y(f) = X(f)e^{j2\pi f t_0} = \Pi\Big(\frac{f}{2}\Big)\,e^{j2\pi f t_0}$$

For non-zero t_0, $y(t)$ has no symmetry, its Fourier transform is a complex-valued function of f. This is because the real and even function $X(f)$ is multiplied by the complex-valued function $e^{j2\pi f t_0}$. The magnitude of this latter function is $1\,\forall f$; the phase equals $2\pi f t_0$. Therefore, $|Y(f)| = |X(f)|$ and $\angle Y(f) = \angle X(f) + 2\pi f t_0$.

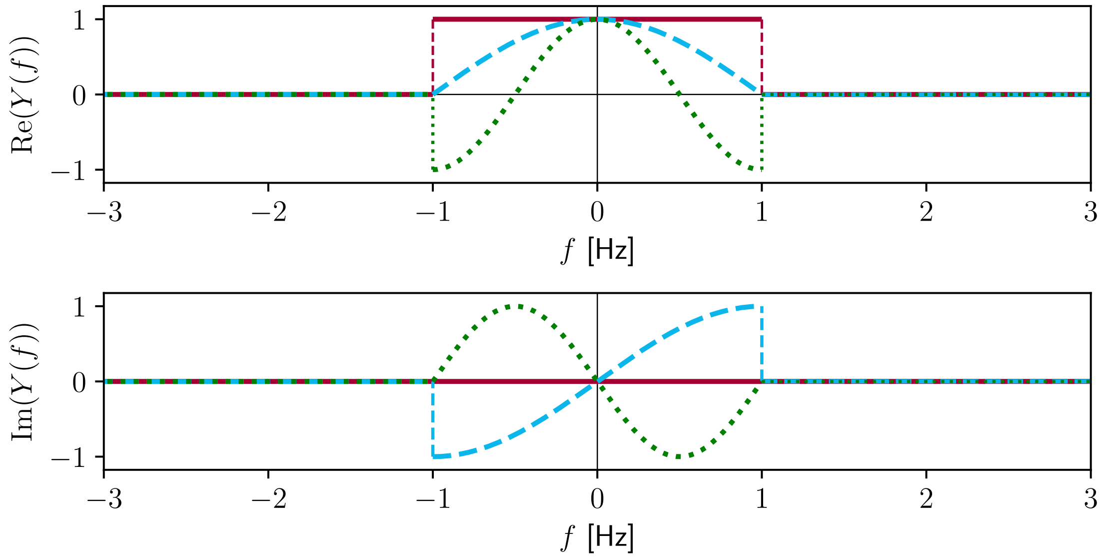
(b) The figure below illustrates $y(t)$ (top row) and its Fourier transform $Y(f)$ for three values of the time shift t_0: 0 s (burgundy), 0.25 s (dashed blue) and -0.5 s (dotted green). As expected, $|Y(f)|$ does not change and equals $|X(f)|$ for all time shifts t_0. For the zero time shift, the phase of $Y(f)$ (and $X(f)$) is $0\ \forall f$: $Y(f)$ is a real-valued even function of f, equal to 1 for $-1 < f < 1$ Hz, equal to 0.5 for $|f| = 1$ Hz and 0 for other f. The non-zero values all occur on the positive real axis in the complex plane, their phases are 0. When $Y(f)$ is 0, its phase is *defined* 0, see Appendix C.



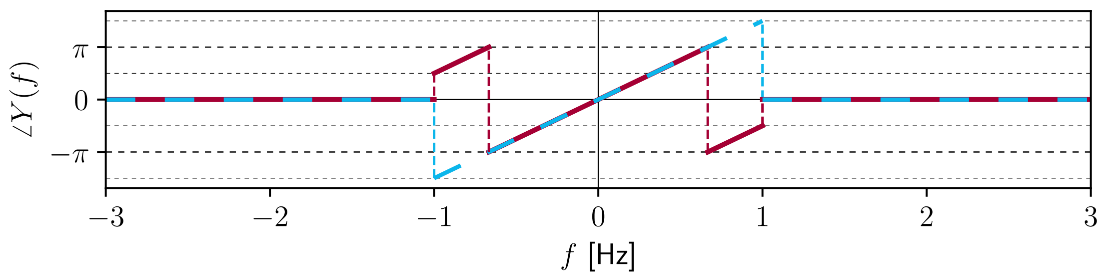
For $t_0 = 0.25$ s, the phase of $Y(f)$ equals $2\pi f(0.25) = \frac{\pi}{2}f$, a line which runs from $-\frac{\pi}{2}$ ($f = -1$ Hz) to $\frac{\pi}{2}$ ($f = 1$ Hz). Here, $y(t)$ will be *ahead* of $x(t)$, a positive phase shift (for positive frequencies).

For $t_0 = -0.5$ s, the phase of $Y(f)$ equals $2\pi f(-0.5) = -\pi f$, a line which runs from π ($f = -1$ Hz) to $-\pi$ ($f = 1$ Hz). Here, $y(t)$ will lag *behind* $x(t)$, a negative phase shift (for positive frequencies).

(c)



(d) For $t_0 = 0.75$ s, the phase of $Y(f)$ equals $2\pi f(0.75) = \frac{3\pi}{2}f$, a line which runs from $-\frac{3\pi}{2}$ ($f = -1$ Hz) to $\frac{3\pi}{2}$ ($f = 1$ Hz). This is illustrated below with the blue dashed line. Because this phase exceeds the limits $[-\pi, \pi]$, see Section 2.3.3, we limit the phase of $Y(f)$ to stay within these limits, as illustrated with the burgundy line.



S | 6.3 Time scale change

S.6-5 Let us name the original (recorded) signal, signal $x(t)$, and the replayed signal, signal $y(t)$. Given that the stored signal is replayed at half the original speed:

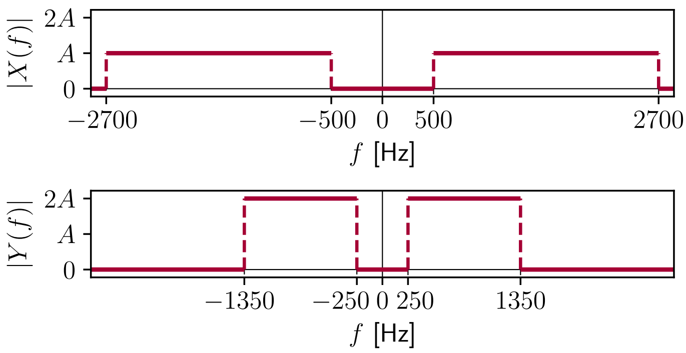
$$y(t) = x\left(\tfrac{1}{2}t\right)$$

Then, using $x(at) \overset{\mathcal{F}}{\leftrightarrow} \frac{1}{|a|}X\left(\frac{f}{a}\right)$, (6.3), if signal $x(t)$ has Fourier transform $X(f)$, then signal $y(t)$ has Fourier transform:

$$Y(f) = \frac{1}{\left|\frac{1}{2}\right|}X\left(\frac{f}{\frac{1}{2}}\right) = 2X(2f)$$

i.e., when time goes twice as slow, frequency runs twice as fast.

Let us now sketch what happens in the frequency domain. Signal $x(t)$ being recorded has an approximately rectangular amplitude spectrum extending from 500 Hz to 2,700 Hz. The top illustration on the right shows amplitude spectrum $|X(f)|$, where the amplitude has been set to A for illustrative purposes. The bottom illustration shows amplitude spectrum $|Y(f)|$. With the amplitude of $X(f)$ being A, the amplitude of $Y(f)$ equals $2A$. The replayed signal will appear to be of much lower frequency than the original (recorded) signal.



S.6-6

(a) Following (5.16), the Fourier transform of $x(t)$ equals:

$$X(f) = \frac{A}{2}\left(\delta(f+f_0)+\delta(f-f_0)\right)$$

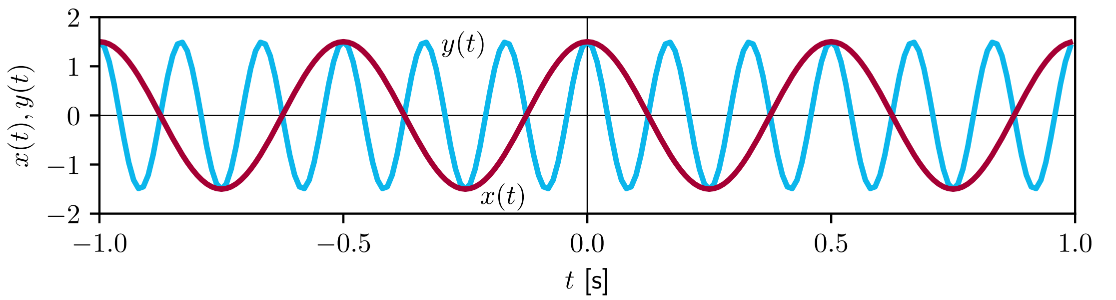
The real, even signal has a real, even Fourier transform. Using the time scale change theorem (6.3):

$$Y(f) = \frac{1}{|a|}X\left(\frac{f}{a}\right) = \frac{1}{a}\frac{A}{2}\left(\delta\left(\frac{f}{a}+f_0\right)+\delta\left(\frac{f}{a}-f_0\right)\right) = \frac{1}{a}\frac{A}{2}\left(\delta\left(\frac{f+af_0}{a}\right)+\delta\left(\frac{f-af_0}{a}\right)\right)$$

where $|a| = a$ because it is given that $a > 0$. We see that the 'running variable' in the Dirac pulses on the right-hand side changed from f to $\frac{f}{a}$. To move back to running variable f in the Dirac pulses, these pulses need to be scaled, see (B.13): $\delta\left(\frac{f+af_0}{a}\right) = a\delta(f+af_0)$ and $\delta\left(\frac{f-af_0}{a}\right) = a\delta(f-af_0)$. Hence:

$$Y(f) = \frac{1}{a}\frac{A}{2}a\left(\delta(f+af_0)+\delta(f-af_0)\right) = \frac{A}{2}\left(\delta(f+af_0)+\delta(f-af_0)\right)$$

(b)



(c) The average power of a sinusoidal signal only depends on its amplitude, hence:

$$P_x = P_y = \frac{A^2}{2}$$

See Example 2.10. While the frequency of $y(t)$ differs from the frequency of $x(t)$, they share the same amplitude. For sinusoidal signals, time scale changes have no effect on their average power.

S|6.4 Time reversal

S.6-7

To compute the amplitude and phase of the Fourier transforms in this exericse, we use:

$$|X(f)| = \sqrt{\mathrm{Re}(X(f))^2+\mathrm{Im}(X(f))^2} \quad (5.6) \qquad \text{and} \qquad \angle X(f) = \arctan\left(\frac{\mathrm{Im}(X(f))}{\mathrm{Re}(X(f))}\right) \quad (5.7)$$

(a) (1) $a(t) = e^{\alpha t}u(-t)$ equals $z(-t)$, with $z(t) = e^{-\alpha t}u(t)$ the function from Example 5.3.

Hence, following the time reversal theorem (6.4), i.e., a special case of the time scale change theorem (6.3): $A(f) = Z(-f)$
Since, as shown in Example 5.3, $Z(f) = \frac{1}{\alpha + j2\pi f}$, we obtain:
$A(f) = \frac{1}{\alpha - j2\pi f}$
This is a complex-valued function of f, similar to $Z(f)$, as $a(t)$ has no symmetry properties, similar to $z(t)$.

(2) $A(f) = \frac{1}{\alpha - j2\pi f}\frac{\alpha + j2\pi f}{\alpha + j2\pi f} = \frac{\alpha + j2\pi f}{\alpha^2 + (2\pi f)^2}$
$\mathrm{Re}(A(f)) = \frac{\alpha}{\alpha^2 + (2\pi f)^2}$ and $\mathrm{Im}(A(f)) = \frac{2\pi f}{\alpha^2 + (2\pi f)^2}$

(3) $|A(f)| = \sqrt{\frac{\alpha^2 + (2\pi f)^2}{(\alpha^2 + (2\pi f)^2)^2}} = \frac{1}{\sqrt{\alpha^2 + (2\pi f)^2}}$ and $\angle A(f) = \arctan\left(\frac{2\pi f}{\alpha}\right)$

(b) (1) For $t < 0$, $b(t) = e^{-\alpha|t|}$ equals $a(t)$, and for $t > 0$, $b(t)$ equals $z(t)$. In fact, $b(t) = a(t) + z(t)$! So, using the linearity theorem (6.1), i.e., $B(f) = A(f) + Z(f)$, we obtain:
$B(f) = \frac{1}{\alpha - j2\pi f} + \frac{1}{\alpha + j2\pi f} = \frac{2\alpha}{(\alpha - j2\pi f)(\alpha + j2\pi f)} = \frac{2\alpha}{\alpha^2 + (2\pi f)^2}$
This function is real and even, simply because $b(t)$ is real and even.

(2) $\mathrm{Re}(B(f)) = \frac{2\alpha}{\alpha^2 + (2\pi f)^2}$ and $\mathrm{Im}(B(f)) = 0$

(3) $|B(f)| = \frac{2\alpha}{\alpha^2 + (2\pi f)^2}$ and $\angle B(f) = 0 \;\forall\, f$

(c) (1) Since $c(t) = e^{-\alpha t}u(t) - e^{\alpha t}u(-t) = z(t) - a(t)$, using the linearity theorem (6.1), we obtain:
$C(f) = Z(f) - A(f) = \frac{1}{\alpha + j2\pi f} - \frac{1}{\alpha - j2\pi f} = \frac{-j4\pi f}{\alpha^2 + (2\pi f)^2}$
This function is imaginary and odd, simply because $c(t)$ is real and odd.

(2) $\mathrm{Re}(C(f)) = 0$ and $\mathrm{Im}(C(f)) = \frac{-4\pi f}{\alpha^2 + (2\pi f)^2}$

(3) $|C(f)| = \frac{4\pi|f|}{\alpha^2 + (2\pi f)^2}$ and $\angle C(f) = 0$ for $f = 0$
$\angle C(f) = \frac{\pi}{2}$ for $f < 0$ and $-\frac{\pi}{2}$ for $f > 0$
due to $\mathrm{Re}(C(f)) = 0$

S | 6.5 Duality

S.6-8

(a) From $\Pi\left(\frac{t}{\tau}\right) \overset{\mathcal{F}}{\leftrightarrow} \tau\,\mathrm{sinc}(\tau f)$, (5.11), and the duality theorem $X(t) \overset{\mathcal{F}}{\leftrightarrow} x(-f)$, (6.5), we know that: $\mathcal{F}\{\tau\,\mathrm{sinc}(\tau t)\} = \Pi\left(\frac{-f}{\tau}\right) = \Pi\left(\frac{f}{\tau}\right)$, which has also been verified in Example 6.1.
Then we obtain for Fourier transform $X(f)$ of signal $x(t) = 4\,\mathrm{sinc}(5t)$:

$$X(f) = 4 \cdot \tfrac{1}{5}\Pi\left(\tfrac{f}{5}\right) = \tfrac{4}{5}\Pi\left(\tfrac{f}{5}\right)$$

Given $y(t) = \frac{1}{2}x(2t)$ and using the time scale change theorem (6.3), $x(at) \overset{\mathcal{F}}{\leftrightarrow} \frac{1}{|a|}X\left(\frac{f}{a}\right)$, we obtain for Fourier transform $Y(f)$:

$$Y(f) = \tfrac{1}{2}\tfrac{1}{|2|}\tfrac{4}{5}\Pi\left(\tfrac{f}{2\cdot 5}\right) = \tfrac{1}{5}\Pi\left(\tfrac{f}{10}\right)$$

As an alternative or as a simple check:
Fourier-transforming $y(t) = \frac{1}{2}4\,\mathrm{sinc}(2\cdot 5t) = 2\,\mathrm{sinc}(10t) = \frac{1}{5}10\,\mathrm{sinc}(10t)$
also results in: $Y(f) = \frac{1}{5}\Pi\left(\frac{f}{10}\right)$

(b) Following a similar approach to that in (a), we obtain for Fourier transform $X(f)$ of signal $x(t) = 3\,\mathrm{sinc}(6t)$:

$$X(f) = 3 \cdot \tfrac{1}{6}\Pi\left(\tfrac{f}{6}\right) = \tfrac{1}{2}\Pi\left(\tfrac{f}{6}\right)$$

Given $y(t) = 4x\left(-\frac{t}{2}\right)$, we obtain for Fourier transform $Y(f)$:

$$Y(f) = 4\frac{1}{\left|-\frac{1}{2}\right|}\frac{1}{2}\Pi\left(\frac{1}{-\frac{1}{2}}\frac{f}{6}\right) = 4\Pi\left(\frac{-f}{3}\right) = 4\Pi\left(\frac{f}{3}\right),$$

where we used the fact that the pulse function is an even function.
As an alternative or as a simple check:
Fourier-transforming $y(t) = 4 \cdot 3\,\mathrm{sinc}\left(-\frac{1}{2}6t\right) = 12\,\mathrm{sinc}(-3t) = 4 \cdot 3\,\mathrm{sinc}(3t)$
where we used the fact that the sinc function is even, also results in: $Y(f) = 4\Pi\left(\frac{f}{3}\right)$

s.6-9

You might remember that you have already obtained time signal $x(t)$ in Exercise e.5-6 using direct substitution in the synthesis equation (5.2), see Solution s.5-6. However, in this exercise, we obtain the same inverse Fourier transform using the duality theorem (6.5).
In Exercise e.6-7(b), the Fourier transform of $b(t) = e^{-\alpha|t|}$ with $\alpha > 0$ was found to be $B(f) = \frac{2\alpha}{\alpha^2 + (2\pi f)^2}$, see Solution s.6-7(b). Using the fact that the Fourier transform is linear, and thus both sides of a pair can be multiplied by a constant, and using the duality theorem $X(t) \overset{\mathcal{F}}{\leftrightarrow} x(-f)$, (6.5), we obtain:

$$x(t) = \frac{2\alpha A}{\alpha^2 + (2\pi t)^2}$$

which is indeed the same result as obtained in Solution s.5-6.

s.6-10

Fourier transform $X(f)$ is shown in Exercise e.6-10. We see that:

$$X(f) = A\Pi\left(\tfrac{f}{7}\right) + A\Pi\left(\tfrac{f}{3}\right)$$

From Exercise e.6-1(c)/Solution s.6-1(c) we know that time signal $x_c(t) = A\Pi\left(\frac{t}{7}\right) + A\Pi\left(\frac{t}{3}\right)$ has Fourier transform $X_c(f) = 7A\,\mathrm{sinc}(7f) + 3A\,\mathrm{sinc}(3f)$. Using the duality theorem (6.5), since $X(f)$ equals $x_c(f)$, the time signal that $X(f)$ originates from, $x(t)$, must equal $X_c(-t)$:

$$x(t) = 7A\,\mathrm{sinc}(-7t) + 3A\,\mathrm{sinc}(-3t) = 7A\,\mathrm{sinc}(7t) + 3A\,\mathrm{sinc}(3t)$$

where in the latter step we used the fact that the sinc function is even.
Hence, time signal $x(t)$ is real and even, as its Fourier transform $X(f)$ is also real and even.

s.6-11

Using $\Pi\left(\frac{t}{\tau}\right) \overset{\mathcal{F}}{\leftrightarrow} \tau\,\mathrm{sinc}(\tau f)$, (5.11), and $\delta(t) \overset{\mathcal{F}}{\leftrightarrow} 1$, (5.14), in combination with the linearity (6.1) and duality (6.5), $X(t) \overset{\mathcal{F}}{\leftrightarrow} x(-f)$, theorems, we obtain for Fourier transform $X(f)$ of signal $x(t) = 1 + 16\,\mathrm{sinc}(4t)$:

$$X(f) = \delta(f) + 4\Pi\left(\tfrac{f}{4}\right)$$

A Dirac pulse occurs in $X(f)$, meaning that this signal has infinite energy. Hence, signal $x(t)$ is a *power* signal with average power:

$$P_x = 1^2 = 1 \text{ W}$$

The average of signal $x(t)$ equals 1; the sinc function becomes irrelevant in this case.
Signal $y(t) = 6\cos(30\pi t) + 2\sin(10\pi t) = 6\cos\big(3(5)(2\pi)t\big) + 2\sin\big(1(5)(2\pi)t\big)$ is a periodic signal with fundamental frequency $f_0 = 5$ Hz. It is thus a *power* signal and its average power equals the sum of the average powers of the individual components:

$$P_y = \frac{6^2}{2} + \frac{2^2}{2} = 20 \text{ W}$$

Alternatively, using $\mathcal{F}\{A\cos(2\pi f_0 t)\} = \frac{A}{2}\big(\delta(f+f_0)+\delta(f-f_0)\big)$, (5.16), and $\mathcal{F}\{A\sin(2\pi f_0 t)\} = j\frac{A}{2}\big(\delta(f+f_0)-\delta(f-f_0)\big)$, (5.17), in combination with the linearity theorem (6.1), we obtain for Fourier transform $Y(f)$:

$$Y(f) = 3\delta(f+15) + 3\delta(f-15) + j\delta(f+5) - j\delta(f-5)$$

Hence, this way, it can also be concluded that $y(t)$ is a *power* signal with average power:

$$P_y = 3^2 + 3^2 + 1^2 + 1^2 = 20 \text{ W}$$

Signal $z(t)$ is also a *power* signal and its average power can be obtained by adding up the average powers of signals $x(t)$ and $y(t)$:

$$P_z = P_x + P_y = 1 + 20 = 21 \text{ W}$$

S|6.7 Modulation

S.6-12 The signals from Exercise *e*.6-1 are modulated onto the carrier signal $p(t) = \cos(12\pi t)$, with frequency $f_0 = 6$ Hz. The modulation theorem, $x(t)\cos(2\pi f_0 t) \overset{\mathcal{F}}{\leftrightarrow} \frac{1}{2}X(f+f_0) + \frac{1}{2}X(f-f_0)$, (6.7) states that one copy of the Fourier transform of the original signal is shifted to f_0 and another one to $-f_0$, and both copies are multiplied by $\frac{1}{2}$.

One can also state that the signals from Exercise *e*.6-1 are multiplied by $p(t)$, where multiplication in time of a signal $x(t)$ with the signal $p(t)$, is equivalent to a convolution in frequency of their Fourier transforms. That is, when $y(t) = x(t)p(t)$, then $Y(f) = X(f) * P(f)$.

(a) $Y_a(f) = 2A\,\text{sinc}\big(2(f+6)\big)\cos\big(5\pi(f+6)\big) + 2A\,\text{sinc}\big(2(f-6)\big)\cos\big(5\pi(f-6)\big)$

(b) $Y_b(f) = 2Aj\,\text{sinc}\big(2(f+6)\big)\sin\big(5\pi(f+6)\big) + 2Aj\,\text{sinc}\big(2(f-6)\big)\sin\big(5\pi(f-6)\big)$

(c) $Y_c(f) = \frac{7A}{2}\text{sinc}\big(7(f+6)\big) + \frac{3A}{2}\text{sinc}\big(3(f+6)\big) + \frac{7A}{2}\text{sinc}\big(7(f-6)\big) + \frac{3A}{2}\text{sinc}\big(3(f-6)\big)$

(d) $Y_d(f) = \frac{A}{2}e^{-j5\pi(f+6)}\Big(3\,\text{sinc}\big(3(f+6)\big) + 5\,\text{sinc}\big(5(f+6)\big)\Big) + \frac{A}{2}e^{-j5\pi(f-6)}\Big(3\,\text{sinc}\big(3(f-6)\big) + 5\,\text{sinc}\big(5(f-6)\big)\Big)$

S.6-13 (a) We will obtain the answer to this question using three different approaches.

Approach 1: Qualitative approach

Signal $x(t) = 10\,\text{sinc}(5t)\cos^2(20\pi t)$ has two components, a sinc component and a cosine-squared component.

The cosine-squared component is periodic, and just goes on and on from $t = -\infty$ to $t = \infty$, fluctuating between 0 and 1. Its energy is infinite, it is a power signal.

The sinc component becomes 0 for large positive or negative t. Recall that $\text{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$, (B.3), and whereas the numerator always remains within ± 1, the denominator becomes $\pm\infty$ for $t = \pm\infty$. This component is therefore an energy signal.

Whether the components are power or energy signals, is, however, not so important. After all, the question is about signal $x(t)$ and not its components. However, having insight into what the components look like is always important.

Clearly, signal $x(t)$ is a multiplication of the sinc and cosine-squared components. Then, what is crucial to note, is that the sinc function, as it becomes smaller and smaller for larger $|t|$, makes the product of the sinc and cosine-squared components also become smaller and smaller. Therefore, signal $x(t)$ has finite energy, and its energy can be computed. Hence, it is an energy signal.

Approach 2: Look at the signal mathematically in the time domain

Using $\cos^2 u = \frac{1}{2}(1+\cos(2u))$, (A.4):

$$x(t) = 10\,\text{sinc}(5t)\cos^2(20\pi t) = 10\frac{\sin(5\pi t)}{5\pi t}\frac{1}{2}\big(1+\cos(40\pi t)\big)$$
$$= 5\frac{\sin(5\pi t)}{5\pi t}\big(1+\cos(40\pi t)\big) = \left(\frac{1}{\pi t}\right)\sin(5\pi t)\,\big(1+\cos(40\pi t)\big)$$

Clearly, we multiply $\frac{1}{\pi t}$ by a term that fluctuates between ± 1 multiplied by a term that fluctuates between 0 and 2. Whereas the result of the product of the (latter) two fluctuations is finite, the first term $\frac{1}{\pi t}$ goes to zero when $|t|$ becomes large. Hence, the signal becomes 0 for larger $|t|$, which makes it an energy signal.

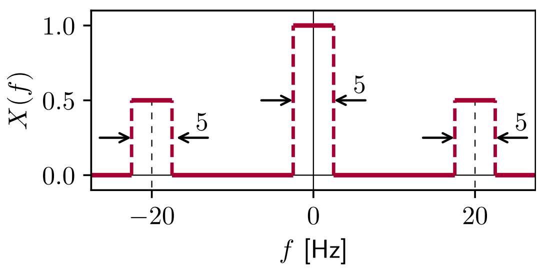
Approach 3: Transform the signal to the frequency domain

$$x(t) = 10\,\text{sinc}(5t)\cos^2(20\pi t) = 10\,\text{sinc}(5t)\,\frac{1}{2}\big(1+\cos(40\pi t)\big)$$
$$= 5\,\text{sinc}(5t) + 5\,\text{sinc}(5t)\cos(40\pi t)$$

From $\Pi\left(\frac{t}{\tau}\right) \overset{\mathcal{F}}{\leftrightarrow} \tau\,\text{sinc}(\tau f)$, (5.11), and the duality theorem, $X(t) \overset{\mathcal{F}}{\leftrightarrow} x(-f)$, (6.5), we know that: $\mathcal{F}\{\tau\,\text{sinc}(\tau t)\} = \Pi\left(\frac{-f}{\tau}\right) = \Pi\left(\frac{f}{\tau}\right)$. Combining this with the linearity (6.1) and modulation , $x_1(t)\cos(2\pi f_0 t) \overset{\mathcal{F}}{\leftrightarrow} \frac{1}{2}X_1(f+f_0)+\frac{1}{2}X_1(f-f_0)$, (6.7), theorems, we obtain for Fourier transform $X(f)$:

$$X(f) = \Pi\left(\frac{f}{5}\right) + \frac{1}{2}\Pi\left(\frac{f+20}{5}\right) + \frac{1}{2}\Pi\left(\frac{f-20}{5}\right)$$

Fourier transform $X(f)$ is illustrated below on the right; it consists of three pulses. Computing the energy of a signal can be done in the frequency domain using Parseval's theorem: $E = \int_{-\infty}^{\infty} |X(f)|^2\,\mathrm{d}f$, (5.19). When this integration yields a number, as is the case for signal $x(t)$, since we simply have to integrate three squared pulses, the signal is an energy signal.



(b) We compute the signal's energy in the frequency domain using the illustration of $X(f)$:

$$E_x = \int_{-\infty}^{\infty}|X(f)|^2\,\mathrm{d}f = \int_{-22.5}^{-17.5}0.5^2\,\mathrm{d}f + \int_{-2.5}^{2.5}1^2\,\mathrm{d}f + \int_{17.5}^{22.5}0.5^2\,\mathrm{d}f = 7.5\text{ J}$$

However, by looking at the illustration, it can also almost directly be seen that:

$$E_x = 5\cdot 0.5^2 + 5\cdot 1^2 + 5\cdot 0.5^2 = \frac{5}{4}+5+\frac{5}{4} = 7.5\text{ J}$$

Computing the signal's energy is much easier in the frequency domain than in the time domain in this case. The lesson learned is that it is always wise to first think about whether computing energy (or power, for that matter) would be easier in the time or frequency domain. With some signals (like a pulse in time), it is much easier to compute energy in the time domain, whereas with other signals (like a sinc function in time), it may be easier to work in the frequency domain.

S.6-14

From $\Pi\left(\frac{t}{\tau}\right) \overset{\mathcal{F}}{\leftrightarrow} \tau\,\text{sinc}(\tau f)$, (5.11), and the duality theorem, $X(t)\overset{\mathcal{F}}{\leftrightarrow}x(-f)$, (6.5), we know that: $\mathcal{F}\{\tau\,\text{sinc}(\tau t)\} = \Pi\left(\frac{-f}{\tau}\right) = \Pi\left(\frac{f}{\tau}\right)$.

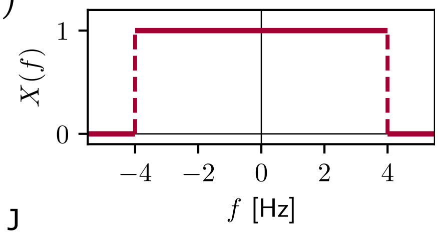
Then we obtain for Fourier transform $X(f)$ of signal $x(t) = 8\,\text{sinc}(8t)$:

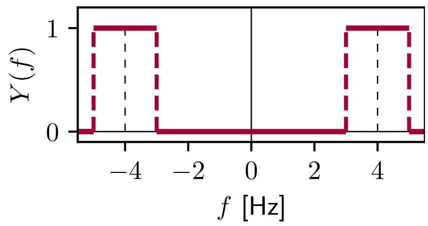
$$X(f) = \Pi\left(\frac{f}{8}\right)$$

Applying the modulation theorem, $x(t)\cos(2\pi f_0 t) \overset{\mathcal{F}}{\leftrightarrow} \frac{1}{2}X(f+f_0)+\frac{1}{2}X(f-f_0)$, (6.7), to signal $y(t) = 4\,\mathrm{sinc}(2t)\cos(8\pi t)$:

$$Y(f) = \tfrac{1}{2}2\Pi\left(\tfrac{f+4}{2}\right) + \tfrac{1}{2}2\Pi\left(\tfrac{f-4}{2}\right) = \Pi\left(\tfrac{f+4}{2}\right) + \Pi\left(\tfrac{f-4}{2}\right)$$

$X(f)$ and $Y(f)$ are illustrated on the right.





Using $E_x = \int_{-\infty}^{\infty} |X(f)|^2\,\mathrm{d}f$, (5.19):

$$E_x = \int_{-4}^{4} 1^2\,\mathrm{d}f = 4-(-4) = 8\text{ J}$$

$$E_y = \int_{-5}^{-3} 1^2\,\mathrm{d}f + \int_{3}^{5} 1^2\,\mathrm{d}f = (-3-(-5)) + (5-3) = 4\text{ J}$$

Using the linearity theorem (6.1): $Z(f) = X(f) + Y(f)$

$$E_z = \int_{-5}^{-4} 1^2\,\mathrm{d}f + \int_{-4}^{-3} 2^2\,\mathrm{d}f + \int_{-3}^{3} 1^2\,\mathrm{d}f + \int_{3}^{4} 2^2\,\mathrm{d}f + \int_{4}^{5} 1^2\,\mathrm{d}f = 1+4+6+4+1 = 16\text{ J}$$

S | 6.8 Convolution

S.6-15

From the illustration of signal $x(t)$: $x(t) = 2\Pi\left(\frac{t}{2}\right)$

Using $\Pi\left(\frac{t}{\tau}\right) \overset{\mathcal{F}}{\leftrightarrow} \tau\,\mathrm{sinc}(\tau f)$, (5.11): $X(f) = 2\cdot 2\,\mathrm{sinc}(2f) = 4\,\mathrm{sinc}(2f)$

From the illustration of signal $y(t)$: $y(t) = \delta(t-6)$

Using $\delta(t) \overset{\mathcal{F}}{\leftrightarrow} 1$ (5.14) and the time shift theorem (6.2): $Y(f) = e^{j2\pi f(-6)} = e^{-j12\pi f}$

Using the convolution theorem (6.8) which states that 'a convolution in the time domain is equivalent to a multiplication in the frequency domain', $x_1(t) * x_2(t) \overset{\mathcal{F}}{\leftrightarrow} X_1(f)X_2(f)$:

$$Z(f) = X(f)Y(f) = 4\,\mathrm{sinc}(2f)\,e^{-j12\pi f}$$

When inverse Fourier-transforming $Z(f) = 2\cdot 2\,\mathrm{sinc}(2f)\,e^{j2\pi f(-6)}$, we obtain:

$$z(t) = 2\Pi\left(\frac{t-6}{2}\right)$$

which makes sense given the illustrations of $x(t)$, a pulse centered at $t = 0$ s, and $y(t)$, a Dirac delta function positioned at $t = 6$ s. See Section 7.3 for an elaboration on convolution with a Dirac delta function.

S | 6.9 Multiplication

S.6-16

A clever solution strategy starts by recognizing that signal $y(t)$ is actually the product of two signals, namely:

$x_1(t) = e^{-\alpha t}$ for $t \geq 0$, or $x_1(t) = e^{-\alpha t}u(t)$, with unit step function $u(t)$, and
$x_2(t) = \cos(2\pi f_0 t)$, such that $y(t) = x_1(t)x_2(t)$.

The multiplication theorem (6.8) can be used, which states that signal $y(t) = x_1(t)\,x_2(t)$ is Fourier-transformed into $Y(f) = X_1(f) * X_2(f)$, i.e., a multiplication in the time domain is equivalent to a convolution in the frequency domain.

$X_1(f)$ is the Fourier transform of $x_1(t)$, as already obtained in Example 5.3: $X_1(f) = \dfrac{1}{\alpha + j2\pi f}$

$X_2(f)$ is the Fourier transform of $x_2(t)$, as already available in (5.16), see also Figure 5.5:
$X_2(f) = \frac{1}{2}\delta(f+f_0) + \frac{1}{2}\delta(f-f_0)$

The requested Fourier transform then becomes:

$$Y(f) = X_1(f) * X_2(f) = \int_{-\infty}^{\infty} X_1(\nu) X_2(f-\nu)\,\mathrm{d}\nu$$
$$= \int_{-\infty}^{\infty} \frac{1}{\alpha + j2\pi\nu}\frac{1}{2}\delta(f+f_0-\nu)\,\mathrm{d}\nu + \int_{-\infty}^{\infty} \frac{1}{\alpha + j2\pi\nu}\frac{1}{2}\delta(f-f_0-\nu)\,\mathrm{d}\nu$$

which may look quite impressive, but can easily be solved using the sifting property of the Dirac delta function, (B.12). Of the first integral, only the value of the first function at $\nu = f + f_0$ remains, and in the second integral at $\nu = f - f_0$:

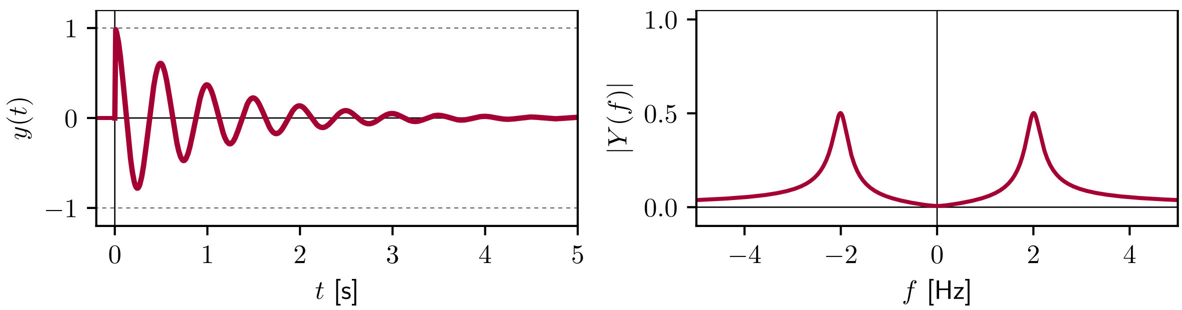
$$Y(f) = \frac{1}{2}\frac{1}{\alpha + j2\pi(f+f_0)} + \frac{1}{2}\frac{1}{\alpha + j2\pi(f-f_0)}$$

The exercise ends here, we are done.

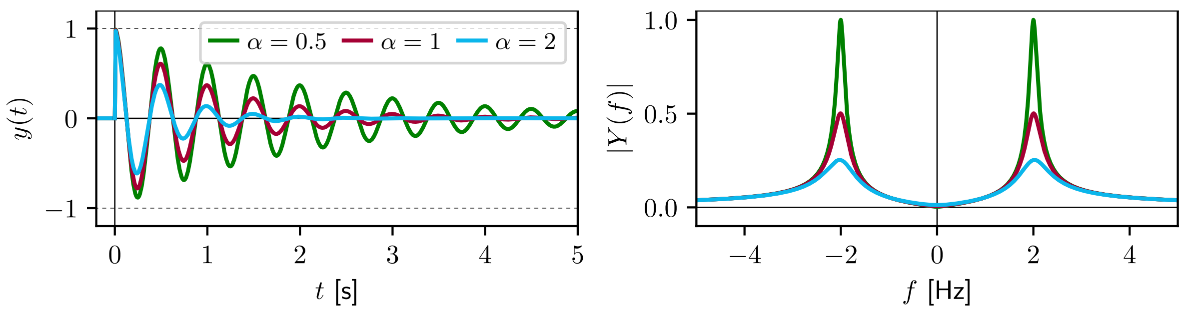
However, let us have a somewhat more in-depth look at the Fourier transform $Y(f)$, using the results from Computer exercise *e*.6-21.

From Figure 5.5 we know that the Fourier transform of a pure (and permanently lasting) cosine consists of two Dirac delta functions, located at $f = -f_0$ and $f = f_0$.

In the figure below we see that the Fourier transform of a *damped* cosine signal $y(t) = e^{-\alpha t}\cos(2\pi f_0 t)$ for $t \geq 0$ (shown at left for $\alpha = 1$ and $f_0 = 2$ Hz) yields an amplitude spectrum consisting of two peaks, which have kind of partly melted (shown at right).



Finally, we demonstrate different amounts of damping. The shape of the peaks – in particular, the height with respect to the width – is driven by the amount of damping, which can be altered by varying time constant $\alpha > 0$. The figure below shows damped cosine signal $y(t)$ and its corresponding amplitude spectrum $|Y(f)|$ for $\alpha = 0.5$ (green), $\alpha = 1$ (burgundy) and $\alpha = 2$ (blue), again with $f_0 = 2$ Hz.

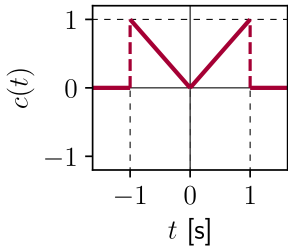


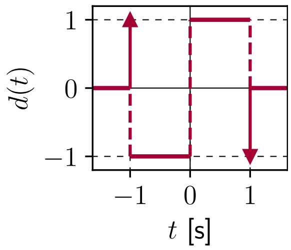
The damped harmonic behavior manifests itself with a mass-spring-damper system as shown in Figure 17.1. Factor α in this exercise corresponds to $\alpha = \zeta\omega_0 = \zeta 2\pi f_0$, with ζ the damping ratio and ω_0 the undamped natural angular frequency in Chapter 17 (see also Figure 17.3). For the same undamped natural frequency ω_0, a larger damping ratio ζ results in a larger value for α, and in a lower peak (α can be shown to be $\alpha = \frac{c}{2m}$, with damping coefficient c and mass m). For zero damping, i.e., $\alpha = 0$, $|Y(f)|$ will return to two Dirac delta functions.

S | 6.10 Differentiation

S.6-17

(a) Signal $x_a(t)$ is (real and) even, so $X_a(f)$ will be real and even.
Suppose we have a signal $c(t)$, shown below on the right, and its derivative $d(t)$, shown directly underneath. Note the unit impulses at $f = -1$ and $f = 1$ Hz as part of signal $d(t)$, cf. Figure 2.5.





Since $\frac{\mathrm{d}}{\mathrm{d}t}c(t) = d(t)$, using the differentiation theorem (6.10), we obtain: $\quad (j2\pi f)C(f) = D(f) \quad$ or $\quad C(f) = \frac{D(f)}{j2\pi f} \qquad (*)$

Derivative $d(t)$ can be mathematically described as:

$$d(t) = \delta(t+1) - \Pi\left(\frac{t+0.5}{1}\right) + \Pi\left(\frac{t-0.5}{1}\right) - \delta(t-1)$$

Using $\Pi\left(\frac{t}{\tau}\right) \overset{\mathcal{F}}{\leftrightarrow} \tau\,\mathrm{sinc}(\tau f)$, (5.11), $\delta(t) \overset{\mathcal{F}}{\leftrightarrow} 1$, (5.14), and the time shift theorem (6.2), we obtain:

$$\begin{aligned} D(f) &= e^{j2\pi f} - \mathrm{sinc}(f)\,e^{j2\pi f 0.5} + \mathrm{sinc}(f)\,e^{-j2\pi f 0.5} - e^{-j2\pi f} \\ &= -\left(e^{-j2\pi f} - e^{j2\pi f}\right) + \mathrm{sinc}(f)\left(e^{-j\pi f} - e^{j\pi f}\right) \end{aligned}$$

Using $j\frac{1}{2}\left(e^{-j\theta} - e^{j\theta}\right) = \sin\theta$, (C.3):

$$D(f) = 2j\underbrace{\sin(2\pi f)}_{\text{odd}} - 2j\underbrace{\underbrace{\mathrm{sinc}(f)}_{\text{even}}\underbrace{\sin(\pi f)}_{\text{odd}}}_{\text{odd}}$$

Note that $\sin(2\pi f)$ is odd, and that $\mathrm{sinc}(f)$ is even and $\sin(\pi f)$ odd, meaning that the product of the latter two is odd. In combination with the sum of two odd functions resulting in an odd function, and $\pm 2j$ being purely imaginary, this means that $D(f)$ is an imaginary-valued and odd function of f. This could also be deduced from the illustration of $d(t)$, as it can be seen that signal $d(t)$ is real and odd, so its Fourier transform must be imaginary-valued and odd.
$C(f)$ can then be computed from $D(f)$ using relation (*) and (B.3):

$$\begin{aligned} C(f) = \frac{1}{j2\pi f}D(f) &= \frac{1}{j2\pi f}2j\sin(2\pi f) - \frac{1}{j2\pi f}2j\,\mathrm{sinc}(f)\sin(\pi f) \\ &= 2\frac{\sin(2\pi f)}{2\pi f} - \mathrm{sinc}(f)\frac{\sin(\pi f)}{\pi f} = 2\,\mathrm{sinc}(2f) - \mathrm{sinc}^2(f) \end{aligned}$$

a real and even function of f, as expected, since $c(t)$ is a real and even function of t.
Since $x_a(t) = c(t+2) + c(t-2)$, using the time shift theorem (6.2), we obtain:

$$X_a(f) = C(f)e^{j4\pi f} + C(f)e^{-j4\pi f} = C(f)\left(e^{-j4\pi f} + e^{j4\pi f}\right)$$

Using $\frac{1}{2}\left(e^{-j\theta} + e^{j\theta}\right) = \cos\theta$, (C.2):

$$X_a(f) = \left(4\underbrace{\mathrm{sinc}(2f)}_{\text{even}} - 2\underbrace{\mathrm{sinc}^2(f)}_{\text{even}}\right)\underbrace{\cos(4\pi f)}_{\text{even}}$$

which is, as expected, a real and even function of f. This results from the fact that $\mathrm{sinc}(2f)$, $\mathrm{sinc}^2(f)$ and $\cos(4\pi f)$ are all even functions, and both the product and the sum of two even functions result in an even function.

(b) Signal $x_b(t)$ is (real and) odd, so $X_b(f)$ will be imaginary and odd.
The Fourier transform of $x_b(t)$ follows from what we have computed in (a).
Since $x_b(t) = c(t+2) - c(t-2)$, using the time shift theorem (6.2), we obtain:

$$X_b(f) = C(f)e^{j4\pi f} - C(f)e^{-j4\pi f} = C(f)\left(e^{j4\pi f} - e^{-j4\pi f}\right)$$

Using $j\frac{1}{2}\left(e^{-j\theta} - e^{j\theta}\right) = \sin\theta$, (C.3):

$$X_b(f) = j\left(4\underbrace{\mathrm{sinc}(2f)}_{\text{even}} - 2\underbrace{\mathrm{sinc}^2(f)}_{\text{even}}\right)\underbrace{\sin(4\pi f)}_{\text{odd}}$$

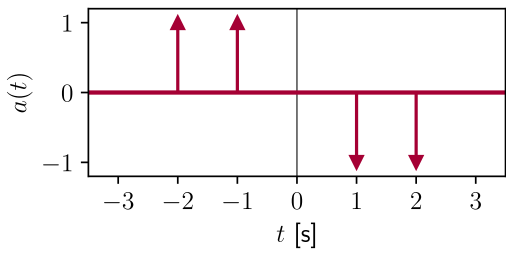
which is, as expected, an imaginary and odd function of f. This results from the fact that $\text{sinc}(2f)$ and $\text{sinc}^2(f)$ are both even, and multiplied by $\sin(4\pi f)$, which is odd. Both multiplications therefore result in an odd function and the sum of two odd functions is again an odd function. The multiplication by j, which is purely imaginary, results in $X_b(f)$ being imaginary-valued and odd.

Note that $\dot{X}(f) = (j2\pi f)X(f)$, with $X(f)$ being the known Fourier transform of signal $x(t)$, can be applied to obtain Fourier transform $\dot{X}(f)$ of signal $\dot{x}(t) = \frac{\mathrm{d}}{\mathrm{d}t}x(t)$, assuming that the Fourier transform $X(f)$ of signal $x(t)$, as well as its derivative $\dot{x}(t) = \frac{\mathrm{d}}{\mathrm{d}t}x(t)$ with Fourier transform $\dot{X}(f)$, exist. However, the deduced relation $X(f) = \frac{\dot{X}(f)}{j2\pi f}$ *cannot* be applied for all signals $x(t)$, to obtain $X(f)$ from the Fourier transform $\dot{X}(f)$ of the signal's derivative $\dot{x}(t) = \frac{\mathrm{d}}{\mathrm{d}t}x(t)$, even when assuming that $\dot{x}(t) = \frac{\mathrm{d}}{\mathrm{d}t}x(t)$, $\dot{X}(f)$ and $X(f)$ all exist. Exercise *e*.6-18 demonstrates for what signals $x(t)$ the latter relation cannot be applied.

s.6-18

(a) The derivative of signal $y_a(t)$, signal $a(t)$, is shown on the right. Since $\frac{\mathrm{d}}{\mathrm{d}t}y_a(t) = a(t)$, using the differentiation theorem (6.10), we obtain the relation:

$$(j2\pi f)Y_a(f) = A(f) \quad \text{or} \quad Y_a(f) = \frac{A(f)}{j2\pi f} \qquad (*)$$



Derivative $a(t)$ can be mathematically described as:

$$a(t) = \delta(t+2) + \delta(t+1) - \delta(t-1) - \delta(t-2)$$

Using $\delta(t) \overset{\mathcal{F}}{\leftrightarrow} 1$, (5.14), and the time shift theorem (6.2), we obtain:

$$A(f) = e^{j2\pi f(2)} + e^{j2\pi f(1)} - e^{j2\pi f(-1)} - e^{j2\pi f(-2)} = e^{j4\pi f} + e^{j2\pi f} - e^{-j2\pi f} - e^{-j4\pi f}$$

Using $j\frac{1}{2}\left(e^{-j\theta} - e^{j\theta}\right) = \sin\theta$, (C.3):

$$A(f) = 2j\sin(4\pi f) + 2j\sin(2\pi f)$$

$Y_a(f)$ can then be computed from $A(f)$ using relation (*) and (B.3):

$$\begin{aligned} Y_a(f) &= \frac{1}{j2\pi f}A(f) = \frac{1}{j2\pi f}2j\sin(4\pi f) + \frac{1}{j2\pi f}2j\sin(2\pi f) = 4\frac{\sin(4\pi f)}{4\pi f} + 2\frac{\sin(2\pi f)}{2\pi f} \\ &= 4\,\text{sinc}(4f) + 2\,\text{sinc}(2f) \end{aligned}$$

Fourier transform $Y_a(f)$ is a real and even function of f, as expected, since $y_a(t)$ is a (real and) even function of t.

(b) Signal $y_a(t)$ can be described as:

$$y_a(t) = \Pi\left(\frac{t}{4}\right) + \Pi\left(\frac{t}{2}\right)$$

Using $\Pi\left(\frac{t}{\tau}\right) \overset{\mathcal{F}}{\leftrightarrow} \tau\,\text{sinc}(\tau f)$, (5.11), and the linearity theorem (6.1), we obtain:

$$Y_a(f) = 4\,\text{sinc}(4f) + 2\,\text{sinc}(2f)$$

which is indeed the same result as that obtained in (a).

(c) Comparing the illustrations of $y_a(t)$ and $y_b(t)$, one can see that:

$$y_b(t) = y_a(t) + 1$$

Using $1 \overset{\mathcal{F}}{\leftrightarrow} \delta(f)$, (5.15), and the linearity theorem (6.1), we obtain Fourier transform in-the-limit:

$$Y_b(f) = Y_a(f) + \delta(f) = 4\,\text{sinc}(4f) + 2\,\text{sinc}(2f) + \delta(f)$$

(d) Using the differentiation theorem (6.10), we obtain:

$$B(f) = (j2\pi f)\,Y_b(f) = (j2\pi f)\left(4\,\mathrm{sinc}(4f) + 2\,\mathrm{sinc}(2f) + \delta(f)\right)$$
$$= (j2\pi f)\left(4\,\mathrm{sinc}(4f) + 2\,\mathrm{sinc}(2f)\right)$$

Note that we deliberately left the $(j2\pi f)$-term at the front of $B(f)$ to preserve the expression of the summation of the two sinc functions. The multiplication of $j2\pi f$ by the Dirac pulse at $f = 0$ is zero.

(e) Fourier transform $A(f)$ was already obtained as an intermediate result in (a): $A(f) = 2j\sin(4\pi f) + 2j\sin(2\pi f)$. This can also be written as:

$$A(f) = 2j(4\pi f)\frac{\sin(4\pi f)}{4\pi f} + 2j(2\pi f)\frac{\sin(2\pi f)}{2\pi f} = (j2\pi f)\left(4\,\mathrm{sinc}(4f) + 2\,\mathrm{sinc}(2f)\right)$$

However, given that the aim of (a) was to obtain the Fourier transform of $y_a(t)$, using the result $Y_a(f)$ and the differentiation theorem (6.10), one would obtain the same expression for $A(f)$:

$$A(f) = (j2\pi f)\,Y_a(f) = (j2\pi f)\left(4\,\mathrm{sinc}(4f) + 2\,\mathrm{sinc}(2f)\right)$$

Comparing $A(f)$ and $B(f)$, one can see that they are exactly the same. This makes sense as $b(t) = \frac{\mathrm{d}}{\mathrm{d}t}y_b(t) = \frac{\mathrm{d}}{\mathrm{d}t}(y_a(t) + 1) = \frac{\mathrm{d}}{\mathrm{d}t}y_a(t) + \frac{\mathrm{d}}{\mathrm{d}t}(1) = \frac{\mathrm{d}}{\mathrm{d}t}y_a(t) = a(t)$.

(f) From the fact that the time-derivatives of both $y_a(t)$ and $y_b(t)$ have the same Fourier transform, i.e., $\mathcal{F}\left\{\frac{\mathrm{d}}{\mathrm{d}t}y_a(t) = a(t)\right\} = \mathcal{F}\left\{\frac{\mathrm{d}}{\mathrm{d}t}y_b(t) = b(t)\right\}$, it follows that $Y_b(f)$ *cannot* be uniquely recovered from $B(f)$. In fact, there are infinitely many possible signals $y_c(t) = y_a(t) + c$ with constant $c \in \mathbb{R}$ of which the derivative would have the exact same Fourier transform. If one would *attempt* to obtain the Fourier transform of $y_b(t)$ from $B(f)$, let us call the obtained result $Z_b(f)$, in the same way as $Y_a(f)$ was successfully obtained from $A(f)$ (as the result in (a) was the same as that in (b)), one would obtain:

$$Z_b(f) = \frac{B(f)}{j2\pi f} = \frac{(j2\pi f)\left(4\,\mathrm{sinc}(4f) + 2\,\mathrm{sinc}(2f)\right)}{j2\pi f} = 4\,\mathrm{sinc}(4f) + 2\,\mathrm{sinc}(2f)$$
$$= Y_a(f) \neq Y_b(f)$$

We know that the constant '1' in $y_b(t)$ results in a Dirac pulse at frequency $f = 0$ in $Y_b(f)$, cf. (5.15), and the differentiation theorem (6.10) multiplies the Fourier transform by $j2\pi f$, hence, at $f = 0$, where $j2\pi f = 0$, we get: $(j2\pi f)\delta(f) = 0\delta(f) = 0$. As a result, one would actually obtain the Fourier transform of signal $y_a(t)$, and *not* that of $y_b(t)$.

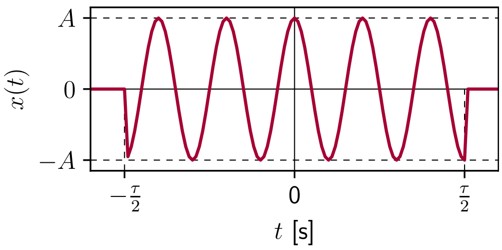
This exercise shows that it is important to realize that one can only use the differentiation theorem (6.10) to obtain the Fourier transform $X(f)$ of a signal $x(t)$ through the Fourier transform $\dot{X}(f)$ of the signal's derivative $\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \dot{x}(t)$, i.e., through the relation $X(f) = \frac{\dot{X}(f)}{j2\pi f}$, if $x(t)$ does *not* include a constant $c \in \mathbb{R} \setminus \{0\}$. The Fourier transform of any signal $x(t)$ that *includes* a constant $c \in \mathbb{R}\setminus\{0\}$ *cannot* be uniquely recovered through the relation $X(f) = \frac{\dot{X}(f)}{j2\pi f}$ when $\dot{X}(f)$ is known.

Note that in Exercise *e*.6-17 our approach with the differentiation theorem only succeeded because both $x_a(t)$ and $x_b(t)$ did *not* include a constant $c \in \mathbb{R} \setminus \{0\}$. If these signals had included such a constant, we would have ended up with the same problem that we encountered in this exercise when attempting to obtain $Y_b(f)$ from $B(f)$.

S|6.c Computer exercises

S.6-19

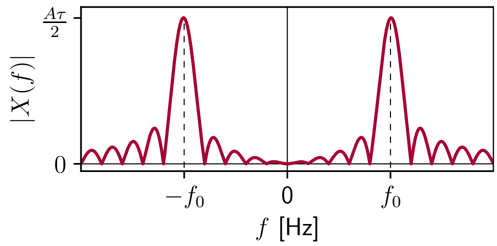
(a) Signal $x(t)$ is illustrated on the right at the top. This illustration has been generated with $\tau = 1$ and $f_0 = 5$ Hz (see basic Python code at the bottom of this solution).



(b) Using $\Pi\left(\frac{t}{\tau}\right) \overset{\mathcal{F}}{\leftrightarrow} \tau\,\mathrm{sinc}(\tau f)$, (5.11), in combination with the modulation theorem $x_1(t)\cos(2\pi f_0 t) \overset{\mathcal{F}}{\leftrightarrow} \frac{1}{2}X_1(f+f_0)+\frac{1}{2}X_1(f-f_0)$, (6.7), we arrive at:

$$X(f) = \frac{A\tau}{2}\left(\mathrm{sinc}(\tau(f+f_0)) + \mathrm{sinc}(\tau(f-f_0))\right)$$

(c) Amplitude spectrum $|X(f)|$ is illustrated on the right at the bottom. This illustration has again been generated with $\tau = 1$ and $f_0 = 5$ Hz.



Note that your own plots of time signal $x(t)$ and amplitude spectrum $|X(f)|$ may look different, depending on the values used for τ and f_0.

Basic Python code:

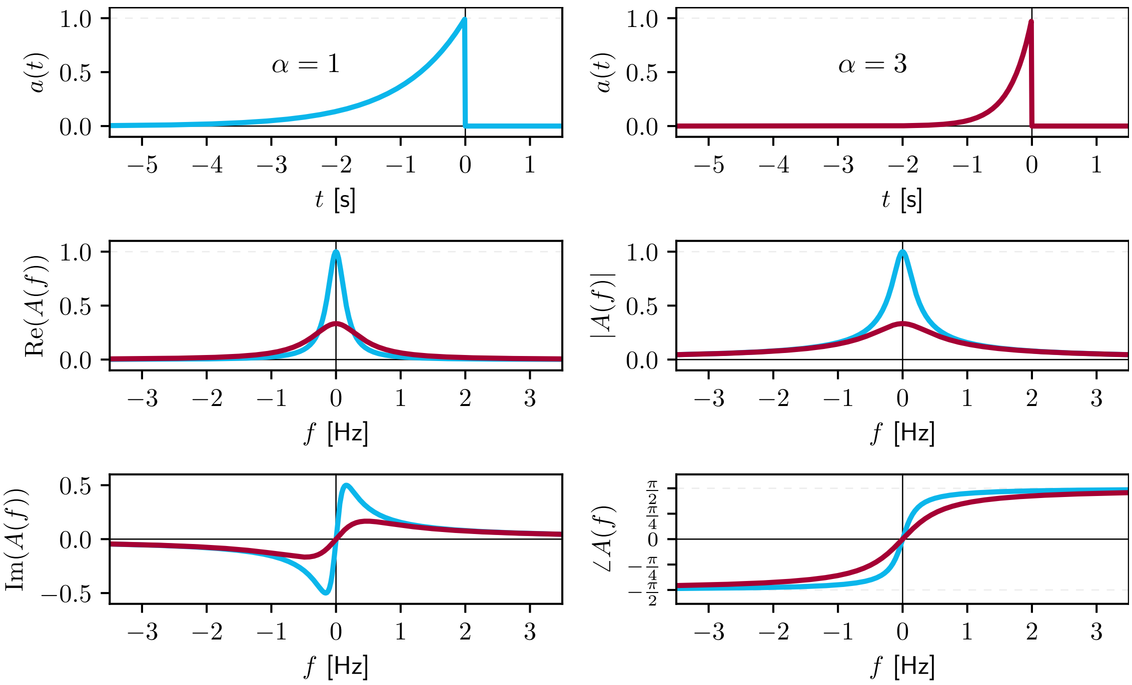
```
import numpy as np
from matplotlib import pyplot as plt

burgundy = [165/255,0/255,52/255]                    # burgundy color

# compute Fourier transform X(f) of sinusoid pulse signal x(t)

# constants
A   = 1 # example value for signal amplitude (not explicitly seen in plot)
tau = 1 # example value for tau
f0  = 5 # example value for cosine carrier frequency [Hz]

# sinusoid pulse signal as function of time
t   = np.linspace(-tau/2-0.25, tau/2+0.25, tau*100+51)  # time steps 0.01 s
p   = np.heaviside(t+tau/2,0) - np.heaviside(t-tau/2,0) # pulse function
cs  = np.cos(2*np.pi*f0*t)                           # cosine signal
x   = A * np.multiply(p, cs)                         # sinusoid pulse signal

# Fourier transform of sinusoid pulse signal
f = np.linspace(-10, 10, 2001) # frequency steps 0.01 Hz from -10 to 10 Hz
X = A * tau / 2 * ( np.sinc(tau*(f+f0)) + np.sinc(tau*(f-f0)) )

# plot of time signal x(t)
plt.figure()
plt.subplot(3,2,1)
plt.plot(t, x, color=burgundy, linewidth=1.5)
plt.xlabel('$t$ [s]'); plt.ylabel('$x(t)$')
# plot of amplitude spectrum |X(f)|
plt.subplot(3,2,3)
plt.plot(f, np.abs(X), color=burgundy, linewidth=1.5)
plt.xlabel('$f$ [Hz]'); plt.ylabel('$|X(f)|$')
```

S.6-20

(a) From Solution *S*.6-7(a), it follows that:

$|A(0)| = \frac{1}{\alpha}, \qquad |A(\pm\infty)| = 0 \qquad \text{and} \qquad \angle A(0) = 0, \quad \angle A(-\infty) = -\frac{\pi}{2}, \quad \angle A(\infty) = \frac{\pi}{2}$

The figure below shows $a(t)$ and $A(f)$, for the values $\alpha = 1$ (blue) and $\alpha = 3$ (burgundy). The top row shows the time signals; the middle and bottom rows the real and imaginary parts of $A(f)$ (at left), and the amplitude and phase spectra (at right). The plots of the real and imaginary parts of $A(f)$ were not explicitly asked for in this exercise, but are added here to provide additional insight.



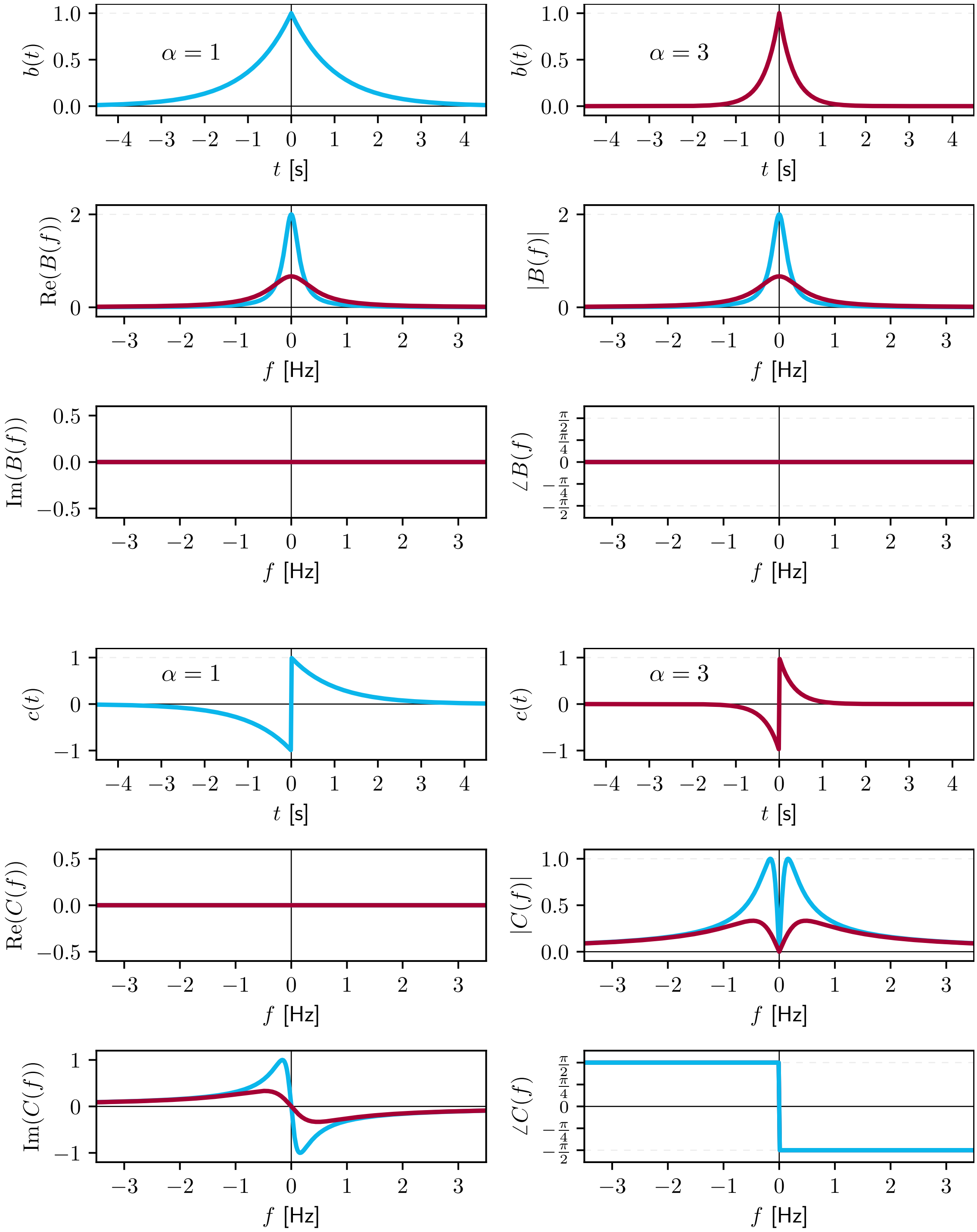
Signal $a(t)$ is mirrored about $t = 0$ s compared to signal $z(t)$. When comparing the amplitude and phase spectra of $a(t)$ with those of $z(t)$ of Example 5.3, one can see that the amplitude spectra are the same, whereas the phase spectrum of $A(f)$ is mirrored about $f = 0$ Hz compared to that of $Z(f)$.

(b) From Solutions *S*.6-7(b) and *S*.6-7(c), it follows that:

$|B(0)| = \frac{2}{\alpha}, \qquad |B(\pm\infty)| = 0 \qquad \text{and} \qquad \angle B(0) = 0, \quad \angle B(-\infty) = 0, \quad \angle B(\infty) = 0$

$|C(0)| = 0, \qquad |C(\pm\infty)| = 0 \qquad \text{and} \qquad \angle C(0) = 0, \quad \angle C(-\infty) = \frac{\pi}{2}, \quad \angle C(\infty) = -\frac{\pi}{2}$

The two figures below show ($b(t)$ and $B(f)$) and ($c(t)$ and $C(f)$), for the values $\alpha = 1$ (blue) and $\alpha = 3$ (burgundy). The top rows show the time signals; the middle and bottom rows the real and imaginary parts of $B(f)/C(f)$ (at left), and the amplitude and phase spectra (at right).



When comparing time signals $b(t)$ and $c(t)$, one can see that for $t < 0$ s, signal $c(t)$ is mirrored about the horizontal axis compared to signal $b(t)$, whereas for $t > 0$ s, the signals are the same. Signal $b(t)$ is an even function of time, whereas signal $c(t)$ is an odd function of time. As a result of these symmetry properties, the imaginary part of $B(f)$ is zero, whereas the real part of $C(f)$ is zero. In turn, as a result, the amplitude spectrum $|B(f)|$ equals the real part of $B(f)$, and the amplitude spectrum $|C(f)|$ equals the absolute value of the imaginary part of $C(f)$.

Basic Python code for signal $a(t)$:

```
import numpy as np
from matplotlib import pyplot as plt

burgundy = [165/255,0/255,52/255]                    # burgundy color

# compute Fourier transform A(f) of exponential decay function a(t)
# and plot real and imaginary parts, and amplitude and phase spectra

# constants
alf1 = 1;          # time constant in exponential decay function (alpha>0)
alf3 = 3;          # larger time constant --> faster decay

# exponential decay function a(t) as function of time
t = np.linspace(-5.5, 5.5, 1101); # time steps 0.01 s from -5.5 s to 5.5 s
at1 = np.heaviside(-t, 0) * np.exp(alf1*t)           # a(t) with alpha = 1
at3 = np.heaviside(-t, 0) * np.exp(alf3*t)           # a(t) with alpha = 3

# Fourier transform A(f) of exponential decay function
f = np.linspace(-5, 5, 1001)  # frequency steps 0.01 Hz from -5 Hz to 5 Hz
p2f = 2 * np.pi * f;                  # 2*pi*f
dal1 = 1 / (alf1*alf1 + p2f*p2f);     # denominator part of A(f) alpha = 1
dal3 = 1 / (alf3*alf3 + p2f*p2f);     # denominator part of A(f) alpha = 3
Ral1 = alf1 * dal1;                   # real part of A(f) alpha = 1
Ial1 = p2f * dal1;                    # imaginary part of A(f) alpha = 1
Ral3 = alf3 * dal3;                   # real part of A(f) alpha = 3
Ial3 = p2f * dal3;                    # imaginary part of A(f) alpha = 3
mXal1 = np.sqrt(dal1);                # magnitude of A(f) alpha = 1
pXal1 = np.arctan2(p2f, alf1);        # phase of A(f) alpha = 1
mXal3 = np.sqrt(dal3);                # magnitude of A(f) alpha = 3
pXal3 = np.arctan2(p2f, alf3);        # phase of A(f) alpha = 3

# plots of time signals (alpha = 1 and alpha = 3)
plt.figure()
plt.subplot(3,2,1)
plt.plot(t, at1, color='b', linewidth=1.5)
plt.xlabel('$t$ [s]'); plt.ylabel('$a(t)$')
plt.subplot(3,2,2)
plt.plot(t, at3, color=burgundy, linewidth=1.5)
plt.xlabel('$t$ [s]'); plt.ylabel('$a(t)$')
# plots of real and imaginary parts of Fourier transforms
plt.subplot(3,2,3)
plt.plot(f, Ral1, color='b', linewidth=1.5)
plt.plot(f, Ral3, color=burgundy, linewidth=1.5)
plt.xlabel('$f$ [Hz]'); plt.ylabel(r'${\rm Re}(A(f))$')
plt.subplot(3,2,5)
plt.plot(f, Ial1, color='b', linewidth=1.5)
plt.plot(f, Ial3, color=burgundy, linewidth=1.5)
plt.xlabel('$f$ [Hz]'); plt.ylabel(r'${\rm Im}(A(f))$')
# plots of magnitude and phase spectra
plt.subplot(3,2,4)
plt.plot(f, mXal1, color='b', linewidth=1.5)
plt.plot(f, mXal3, color=burgundy, linewidth=1.5)
plt.xlabel('$f$ [Hz]'); plt.ylabel('$|A(f)|$')
```

```
ax = plt.subplot(3,2,6)
plt.plot(f, pXal1, color='b', linewidth=1.5)
plt.plot(f, pXal3, color=burgundy, linewidth=1.5)
plt.xlabel('$f$ [Hz]'); plt.ylabel('$\\angle A(f)$')

# the remainder of the code for signals b(t) and c(t) is highly similar
# (except for differences between the signals themselves) to the code
# above for a(t) and is therefore omitted to avoid excessive repetition
```

The results of the script were shown and discussed in the extended discussion of Solution *S*.6-16. *S*.6-21

Basic Python code:

```
import numpy as np
from matplotlib import pyplot as plt

burgundy = [165/255,0/255,52/255]                # burgundy color

# compute Fourier transform Y(f) of damped cosine signal y(t)
# with different amounts of damping

# constants
f0  = 2          # cosine carrier frequency [Hz]
a1  = 1          # time constant in exponential damping function (alpha>0)
a2  = 2          # larger time constant --> more damping
a05 = 0.5        # smaller time constant --> less damping

# damped cosine signal as function of time
t = np.linspace(0, 5, 501)             # time steps 0.01 s from 0 s to 5 s
x1a1  = np.exp(-a1 * t) * np.heaviside(t, 0)     # x_1(t) with alpha = 1
x1a2  = np.exp(-a2 * t) * np.heaviside(t, 0)     # x_1(t) with alpha = 2
x1a05 = np.exp(-a05 * t) * np.heaviside(t, 0)    # x_1(t) with alpha = 0.5
x2    = np.cos(2 * np.pi * f0 * t)  # x_2(t) undamped cosine
yt1   = np.multiply(x1a1, x2)         # y(t) damped cosine with alpha = 1
yt2   = np.multiply(x1a2, x2)         # y(t) damped cosine with alpha = 2
yt05  = np.multiply(x1a05, x2)        # y(t) damped cosine with alpha = 0.5

# Fourier transform of damped cosine signal
f = np.linspace(-5, 5, 1001) # frequency steps 0.01 Hz from -5 Hz to 5 Hz
Yf1   =  (0.5 * np.divide(np.ones(len(f)), a1+2j*np.pi*(f+f0))
          +  0.5 * np.divide(np.ones(len(f)), a1+2j*np.pi*(f-f0)))
Yf2   =  (0.5 * np.divide(np.ones(len(f)), a2+2j*np.pi*(f+f0))
          + 0.5 * np.divide(np.ones(len(f)), a2+2j*np.pi*(f-f0)))
Yf05 =  (0.5 * np.divide(np.ones(len(f)), a05+2j*np.pi*(f+f0))
          + 0.5 * np.divide(np.ones(len(f)), a05+2j*np.pi*(f-f0)))

# plot of time signal
plt.figure()
plt.subplot(3,2,1)
plt.plot([-0.2, 0.0], [0.0, 0.0], color='g', linewidth=1.5)
plt.plot([-0.2, 0.0], [0.0, 0.0], color=burgundy, linewidth=1.5)
plt.plot([-0.2, 0.0], [0.0, 0.0], color='b', linewidth=1.5)
plt.plot(t, yt05, color='g', linewidth=1.5, label='$\alpha=0.5$')
```

```
plt.plot(t, yt1, color=burgundy, linewidth=1.5, label='$\alpha=1$')
plt.plot(t, yt2, color='b', linewidth=1.5, label='$\alpha=2$')
plt.xlabel('$t$ [s]'); plt.ylabel('$y(t)$')
# plot of amplitude spectrum
plt.subplot(3,2,2)
plt.plot(f, np.abs(Yf05), color='g', linewidth=1.5)
plt.plot(f, np.abs(Yf1), color=burgundy, linewidth=1.5)
plt.plot(f, np.abs(Yf2), color='b', linewidth=1.5)
plt.xlabel('$f$ [Hz]'); plt.ylabel('$|Y(f)|$')
```

7

Convolution

Exercises

e|**7.2** Properties of convolution

Prove the commutative property of convolution as defined in (7.3). *e.*7-1

Consider the convolution of signals *e.*7-2

$$x(t) = u(t) \quad \text{and} \quad h(t) = 2\Pi\left(\frac{t-3}{2}\right)$$

from Section 7.1.

Compute the convolution analytically, i.e., through defining and solving the convolution integral, but here for the case where $h(\tau)$ 'remains where it is' and $x(\tau)$ is time-reversed and shifted. Thus, show the commutative property of convolution (7.3) through evaluating:

$$y(t) = x(t) * h(t) = h(t) * x(t) = \int_{\tau=-\infty}^{\infty} h(\tau)x(t-\tau)\,\mathrm{d}\tau$$

This is illustrated in Figure 7.5. We should arrive at the same answer as in Section 7.1, (7.2).

e|**7.5** Examples

Consider signal *e.*7-3

$$x(t) = 2 + 4\,\mathrm{sinc}(4t)\big(1 + 2\cos(16\pi t)\big).$$

Signal $y(t)$ is defined as the derivative of $x(t)$, i.e., $y(t) = \frac{\mathrm{d}}{\mathrm{d}t}x(t)$.

(a) Determine and sketch the Fourier transform of $x(t)$. Describe its properties.

(b) Determine and sketch the Fourier transform of $y(t)$. Describe its properties.

Solutions

s|7.2 Properties of convolution

s.7-1

From (7.3): $$y(t) = x(t) * h(t) = \int_{\tau=-\infty}^{\infty} x(\tau)h(t-\tau)\,\mathrm{d}\tau$$

Define $\xi = t-\tau$, then $\tau = t-\xi$, $\mathrm{d}\tau = -\,\mathrm{d}\xi$ (note that τ is the 'running variable' and within the integral t is a constant). When $\tau \to -\infty$ then $\xi \to \infty$, and when $\tau \to \infty$ then $\xi \to -\infty$. We get:

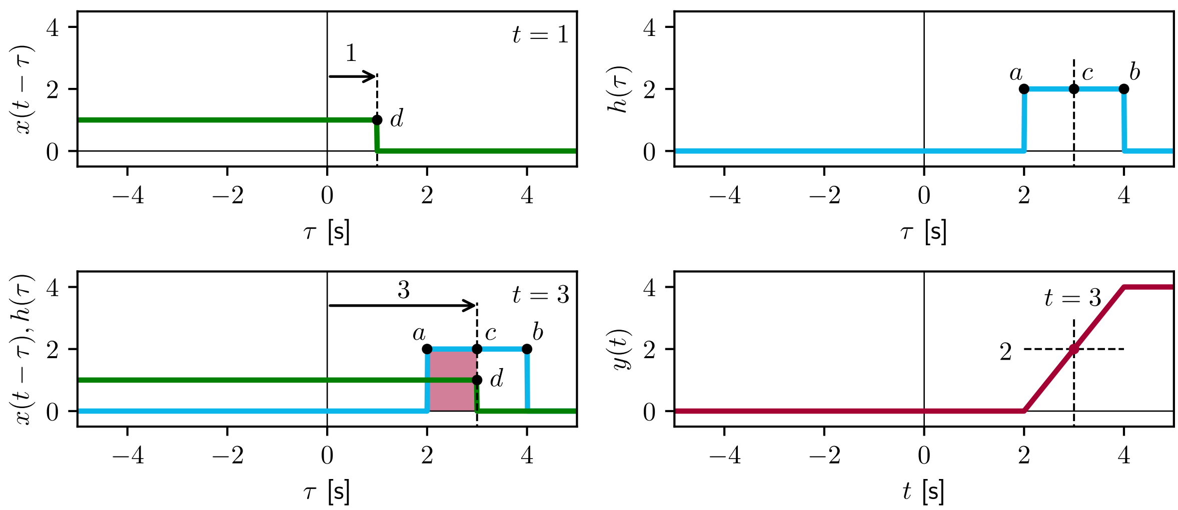
$$y(t) = \int_{\xi=\infty}^{-\infty} x(t-\xi)h(\xi)\,\mathrm{d}(-\xi) = -- \int_{\xi=-\infty}^{\infty} h(\xi)x(t-\xi)\,\mathrm{d}\xi = h(t) * x(t) \qquad \textbf{qed}$$

where in the last step we substituted $\xi = \tau$.

s.7-2

In the first step, we consider both functions as a function of running variable τ, $x(\tau)$ and $h(\tau)$. Function $h(\tau)$ is a pulse (amplitude 2, width 2 s) centered at $\tau = 3$ s, it is non-zero for 2 s $\leq \tau \leq 4$ s. Function $x(\tau)$ is a unit step function. Both functions are illustrated in Figure 7.1.
In the second step, we swap signal $x(\tau)$ around $\tau = 0$ s, yielding $x(-\tau)$, and in the third step we shift this signal $x(-\tau)$ by t, yielding $x(t-\tau)$. What results is a time-reversed step function $u(t-\tau)$ with amplitude 1. It equals 1 for $\tau < t$, 0.5 at $\tau = t$ (point (d)) and 0 for $\tau > t$. So, this function is zero for $\tau > t$ and point (d) lies at $\tau = t$. This result is illustrated in the figure below (top row) for time shift $t = 1$ s, with the swapped and shifted unit step function $x(t-\tau) = u(t-\tau)$ in green at left, and the pulse function $h(\tau)$ in blue at right.
The fourth step is broken down into three parts. First, we note that for any $t < 2$ s, the product of $h(\tau)$ and $x(t-\tau)$ is zero, and so the integral of this product over τ will be zero. This is because for $t < 2$ s, point (d) lies to the left of point (a), the 'left flank' of the pulse function $h(\tau)$. We conclude that $y(t) = 0$ for $t < 2$ s.
In the second part, at $t = 2$ s, function $x(t-\tau)$ enters into overlap with the pulse function. It covers the range from $t \geq 2$ s to $t \leq 4$ s, hence 2 s $\leq t \leq 4$ s. The figure below (bottom row) shows the situation for time shift $t = 3$ s, where both the swapped and shifted unit step function $x(t-\tau) = u(t-\tau)$ and the pulse function $h(\tau)$ are shown at left, and convolution output $y(t)$ is shown in burgundy at right. For this range of time shifts t we obtain the general expression:

$$y(t) = \int_{2}^{t} 2\,\mathrm{d}\tau = 2(t-2) \qquad \text{for} \qquad 2 \leq t \leq 4$$



In the third part, for $t > 4$ s, the unit step function $u(t-\tau)$ fully overlaps the pulse function. The boundaries of the integral where the product of both functions is non-zero are the left edge of the pulse at $\tau = 2$ s (point (a)) and its right edge at $\tau = 4$ s (point (b)):

$$y(t) = \int_2^4 2\,\mathrm{d}\tau = 4 \qquad \text{for} \quad t > 4$$

Thereby, we have also covered the fifth step of evaluating the convolution integral for every value of time shift t.
The resulting function $y(t)$ is shown in Figure 7.5, and is given by:

$$y(t) \;=\; \begin{cases} 0 & \text{for} \quad t < 2 \\ 2(t-2) & \text{for} \quad 2 \le t \le 4 \\ 4 & \text{for} \quad t > 4 \end{cases}$$

This answer is indeed the same as (7.2). This exercise has shown the commutative property of the convolution integral, (7.3), as discussed in Section 7.2.

s|7.5 Examples

S.7-3

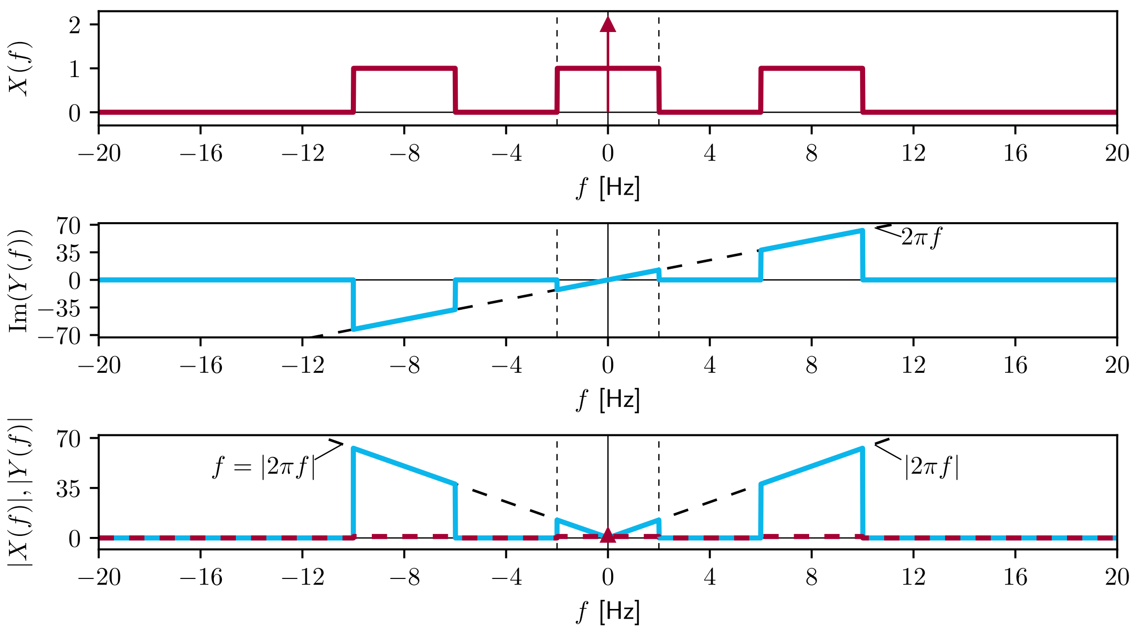
(a) Signal $x(t) = 2 + 4\,\mathrm{sinc}(4t)\big(1 + 2\cos(16\pi t)\big)$ can be written as:

$$x(t) = 2 + x_a(t)\,x_b(t) \quad \text{with} \quad x_a(t) = \mathrm{sinc}(4t) \quad \text{and} \quad x_b(t) = \big(1 + 2\cos(16\pi t)\big)$$

Using the linearity theorem (6.1), the multiplication theorem (6.9), and relationship (7.6), marking the convolution of a Fourier transform with a Dirac function in the frequency domain, we quickly obtain:

$$\begin{aligned} X(f) &= 2\delta(f) + \underbrace{\Pi\Big(\frac{f}{4}\Big)}_{X_a(f)} * \underbrace{\big(\delta(f) + \delta(f+8) + \delta(f-8)\big)}_{X_b(f)} \\ &= 2\delta(f) + \Pi\Big(\frac{f}{4}\Big) + \Pi\Big(\frac{f+8}{4}\Big) + \Pi\Big(\frac{f-8}{4}\Big) \end{aligned}$$

where we used (5.15), (5.16) and (6.13).
Since $x(t)$ is real and even, $X(f)$ is real and even. The figure below shows $X(f)$ in burgundy at top.

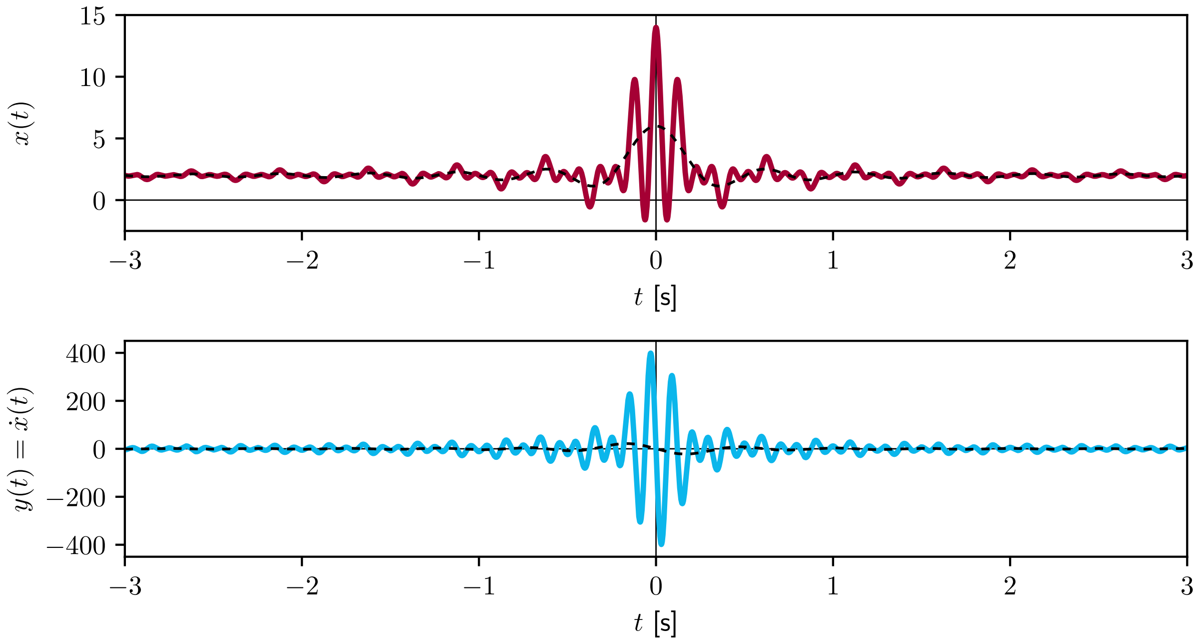


(b) Use the differentiation theorem (6.10), $Y(f) = (j2\pi f)^1 X(f) = (j2\pi f)X(f)$:

$$
\begin{aligned}
Y(f) &= j(2\pi f)\left(2\delta(f) + \Pi\left(\frac{f}{4}\right) + \Pi\left(\frac{f+8}{4}\right) + \Pi\left(\frac{f-8}{4}\right)\right) \\
&= j\underbrace{(2\pi f)}_{\text{odd}}\underbrace{\left(\Pi\left(\frac{f}{4}\right) + \Pi\left(\frac{f+8}{4}\right) + \Pi\left(\frac{f-8}{4}\right)\right)}_{\text{even}} \qquad (*)
\end{aligned}
$$

Note that we deliberately left the $j(2\pi f)$-term at the front of equation (*) such that we can quickly check the properties of $Y(f)$. The multiplication of $j(2\pi f)$ by the Dirac pulse at $f = 0$ is zero. In fact, $Y(f = 0) = j(2\pi 0)X(0) = 0$, no matter the value of $X(0)$. We see that $Y(f)$ is an imaginary-valued (we multiply a real-valued function of f by j), odd (we multiply an odd function of f by an even function of f) function of f. Signal $y(t)$ must then be a real, odd function of time. This means that when we would integrate $y(t)$ from $t = -\infty$ to ∞, (5.8), the result would be zero. $Y(f)$ is shown in blue in the middle plot on the previous page, illustrating only its imaginary part, as its real part equals zero.

Differentiating a signal $x(t)$ means that in the frequency domain we multiply its Fourier transform $X(f)$ by $j2\pi f$. The magnitudes of the Fourier transform at frequencies larger than $|f| = \frac{1}{2\pi}$ Hz increase, the magnitudes at frequencies smaller than $|f| = \frac{1}{2\pi}$ Hz decrease. To illustrate this point further, the figure on the previous page also shows $|X(f)|$ (dashed burgundy) and $|Y(f)|$ (solid blue) at bottom. Because of differentiation, the magnitude of $Y(f)$ is so large that $|X(f)|$ can hardly be distinguished.
This effect, that a differentiated signal typically has (much) higher frequency content than the signal being differentiated, is further illustrated in the figure below, which shows $x(t)$ in burgundy at top and its derivative $y(t)$ in blue at bottom. In both plots, the black dashed curve shows the contribution of the frequencies between -2 and 2 Hz in $X(f)$ and $Y(f)$ to, respectively, $x(t)$ and $y(t)$. Clearly, after differentiation, this contribution becomes negligible.



The effects of integrating a signal on its spectrum and appearance are opposite to the effects of differentiating a signal as shown here; these effects are not discussed further.

8

Finite signal duration, leakage and windowing

Exercises

e|8.c Computer exercises

e.8-1

Consider the cosine signal

$$x(t) = \cos(2\pi f_0 t)$$

with a frequency of $f_0 = 2$ Hz, and unit amplitude.

Demonstrate the effects of time-windowing signal $x(t)$ using a rectangular window by showing Fourier transform $X_w(f)$ of time-windowed signal $x_w(t)$ for the following three window lengths:

(a) $T = 1$ s (b) $T = 2$ s (c) $T = 10$ s

e.8-2

In Chapter 8, both the basic rectangular window and the Hann window were covered, of which the latter is a member of a family of raised cosine windows. In this exercise, you will make a comparison of the effects these two windows have on the time-domain signal, as well as on the spectral representation of the signal.

Consider again signal

$$x(t) = \tau \operatorname{sinc}^2(\tau t)$$

and its Fourier transform

$$X(f) = \Lambda\left(\frac{f}{\tau}\right)$$

with $\tau = 1$, as shown in Figure 8.2.

(a) Create and plot signal $x(t)$. You can approximate this continuous function of time by using a small step size of 0.01 s. Consistent with Figure 8.2, start at $t = -3$ s and end at $t = 3$ s (included as last time point, hence, closed interval).

(b) Create and plot the signal's Fourier transform $X(f)$ from the analytical expression derived in Example 7.3. Approximate the continuous function of frequency $X(f)$ by using a small step size of 0.01 Hz. Also consistent with Figure 8.2, start at $f = -6$ Hz and end at $f = 6$ Hz (included as last frequency value).

(c) Define two window functions $w(t)$, one for the rectangular window and one for the Hann window, symmetric about $t = 0$ s and with duration $T = 3$ s. Set the Fourier transforms $W(f)$ of the two windows as well.

(d) Apply the two windows separately to signal $x(t)$. Obtain the two windowed signals in the time domain, i.e., $x_w(t)$, as well as in the frequency domain, i.e., $X_w(f)$.
Hint: in Python, one can use the function `numpy.convolve` to approximate the convolution integral, $X_w(f) = W(f) * X(f)$ in the frequency domain, by numerical integration. Multiply the result by 0.01 Hz to correct for the step size, cf. (F.3).

(e) Plot the two windowed time-domain signals $x_w(t)$ in the same graph as original signal $x(t)$. Compare the effects of the two windows.

(f) Plot the Fourier transforms of the two windowed signals $X_w(f)$ in the same graph as the Fourier transform of the original signal $X(f)$. Again, compare the effects of the two windows, but now in the frequency domain.

Solutions

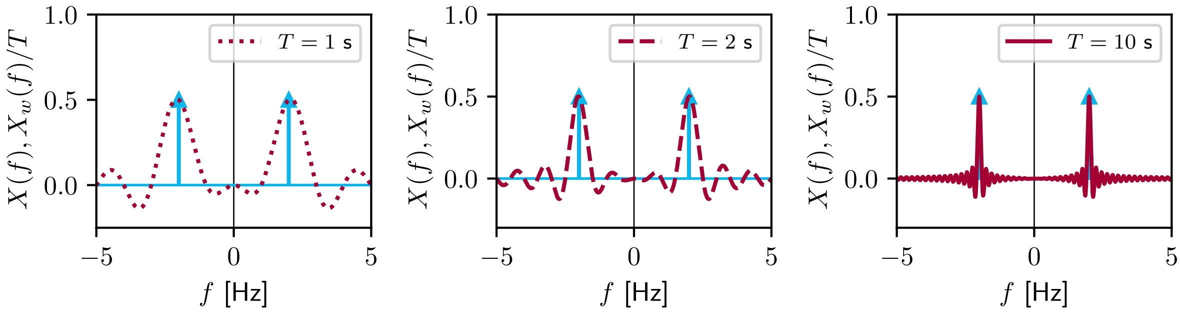
s|8.c Computer exercises

S.8-1

(a)-(c) The mathematical expression for Fourier transform $X_w(f)$ of time-windowed signal $x_w(t)$ is given by (8.3). For $x(t) = \cos(2\pi f_0 t)$ with $f_0 = 2$ Hz, we obtain:

$$X_w(f) = W(f) * X(f) = \frac{T}{2}\Big(\mathrm{sinc}\big(T(f+2)\big) + \mathrm{sinc}\big(T(f-2)\big)\Big)$$

In the figure below, we divide this expression in terms of amplitude by T, hence showing $\frac{X_w(f)}{T}$ (in burgundy), as to have all three cases in analogy with Fourier transform $X(f)$ of the originally infinite-duration signal $x(t)$, which consists of two Dirac delta functions with weights equal to $\frac{1}{2}$ (shown in blue).



For the window lengths of $T = 1$ s and $T = 2$ s, the sinc function has a substantial main-lobe width, and considerable leakage occurs. For a window length of $T = 10$ s, hardly any leakage occurs. Using a larger window length is the only true way of reducing effects of leakage.

Basic Python code:

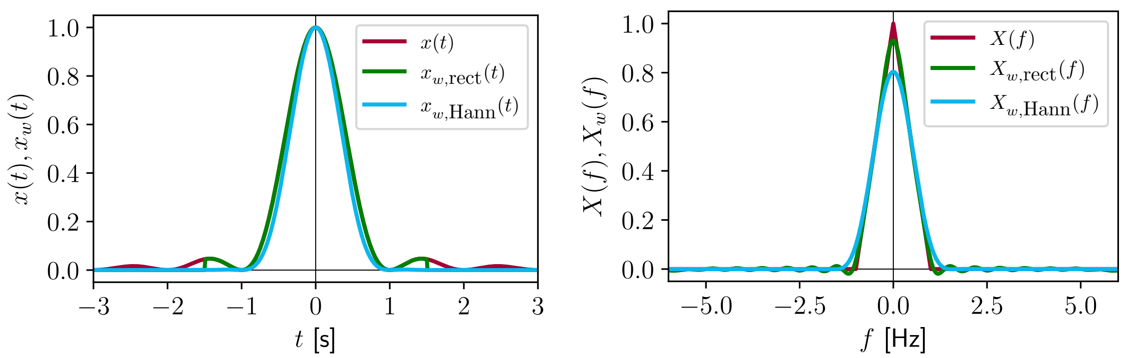
```
import numpy as np
from matplotlib import pyplot as plt

burgundy = [165/255,0/255,52/255]                    # burgundy color

# demonstration of the effect of time-windowing cosine signal x(t)
# with x(t) in continuous time and X(f) in continuous frequency
# show time-windowed Fourier transform Xw(f) for different window lengths

# x(t)  = cos(2 pi f0 t)          this is an even function (unit amplitude)
# X(f)  = 1/2 delta(f+f0) + 1/2 delta(f-f0)    so, Fourier transform is real

# Fourier transform of infinite-duration signal x(t)
# (analytical) Fourier transform (in-the-limit)
f0 = 2                                              # cosine frequency [Hz]
fd = ([-f0, f0])                                    # two Dirac delta functions
Xf = ([0.5, 0.5])                                   # weight 1/2

# show leakage due to three different (rectangular) window lengths
# apply a time window w(t) = Pi(t/T) with T = 1, 2 and 10 [s]
T1 = 1; T2 = 2; T10 = 10                            # window duration [s]

# Fourier transform of continuous-time rectangular windowed cosine
# Xwf has been divided by T (amplitude scaling)
```

```
# to maintain equivalence with infinite-duration cosine signal
f = np.linspace(-6, 6, 1201)  # frequency steps 0.01 Hz from -6 Hz to 6 Hz
Xwf1  = 0.5 * (np.sinc(T1 *(f+f0)) + np.sinc(T1 *(f-f0)))
Xwf2  = 0.5 * (np.sinc(T2 *(f+f0)) + np.sinc(T2 *(f-f0)))
Xwf10 = 0.5 * (np.sinc(T10*(f+f0)) + np.sinc(T10*(f-f0)))

# plot Fourier transform Xw(f) of time-windowed signal x_w(t)
plt.figure()
plt.subplot(3,3,1)                              # T = 1 s
plt.axhline(0, color='b', linewidth=1)
plt.stem(fd, Xf, basefmt=' ', linefmt='b', markerfmt='b^')
plt.plot(f, Xwf1, color=burgundy, linestyle=':', label='$T=1$ s')
plt.xlabel('$f$ [Hz]'); plt.ylabel('$X(f), X_w(f)/T$')
plt.subplot(3,3,2)                              # T = 2 s
plt.axhline(0, color='b', linewidth=1)
plt.stem(fd, Xf, basefmt=' ', linefmt='b', markerfmt='b^')
plt.plot(f, Xwf2, color=burgundy, linestyle='--', label='$T=2$ s')
plt.xlabel('$f$ [Hz]'); plt.ylabel('$X(f), X_w(f)/T$')
plt.subplot(3,3,3)                              # T = 10 s
plt.axhline(0, color='b', linewidth=1)
plt.stem(fd, Xf, basefmt=' ', linefmt='b', markerfmt='b^')
plt.plot(f, Xwf10, color=burgundy, label='$T=10$ s')
plt.xlabel('$f$ [Hz]'); plt.ylabel('$X(f), X_w(f)/T$')
```

S.8-2

(a)-(f) Remember that applying the window implies multiplication in the time domain, $x_w(t) = w(t)x(t)$, and convolution in the frequency domain, $X_w(f) = W(f) * X(f)$. The convolution integral is approximated by numerical integration, with a step size of 0.01 Hz.

The figure below at left shows the original signal $x(t)$ in burgundy, the rectangular-windowed signal in green, and the Hann-windowed signal in blue, the latter two with a window duration of $T = 3$ s. One can clearly see the abrupt start and end of the rectangular window: the green line suddenly jumps up at $t = -1.5$ s and suddenly drops to zero at $t = 1.5$ s. The Hann window, instead, gradually starts and ends (smooth transition).



The figure above at right shows the Fourier transform of the original signal $X(f)$ in burgundy, being the triangular function. The Fourier transforms of the rectangular-windowed signal and of the Hann-windowed signal are shown in green and blue, respectively. The Hann window gives rise to a widening (and lowering) of the peak, but it does not show 'wobbles' in the tail, thanks to lower side lobes in $W(f)$, compared to the rectangular window, see Figure 8.8.

On purpose we used (real and) even signals in this exercise, both $x(t)$ and $w(t)$, such that the Fourier transforms are real-valued and even functions of frequency.

Basic Python code:

```
import numpy as np
from matplotlib import pyplot as plt

burgundy = [165/255,0/255,52/255]      # burgundy color

# time-domain signal x(t)
t  = np.linspace(-3, 3, 601)           # time steps 0.01 s from -3 s to 3 s
xt = np.sinc(t)**2                     # x(t) = sinc^2(t)

# Fourier transform X(f) of signal x(t)
f  = np.linspace(-6, 6, 1201) # frequency steps 0.01 Hz from -6 Hz to 6 Hz
Xf = ( (np.heaviside(f+1,0) - np.heaviside(f,0)) * (f+1) # X(f)=\Lambda(f)
     + (np.heaviside(f,0) - np.heaviside(f-1,0)) * (-(f-1)) )

# define rectangular and Hann windows in time domain
T = 3                             # window duration [s]
wt_rect = np.heaviside(t+T/2, 0) - np.heaviside(t-T/2, 0)     # rectangular
wt_Hann = wt_rect * ( 0.5 + 0.5 * np.cos(2*np.pi*t/T) )       # Hann
# set Fourier transforms of windows
Wf_rect = T * np.sinc(T*f)        # Fourier transform of rectangular window
Wf_Hann = ( T/2 * np.sinc(T*f)    # Fourier transform of Hann window
            + T/4 * ( np.sinc(T*(f + 1/T)) + np.sinc(T*(f - 1/T)) ) )

# apply windows in time domain
xwt_rect = wt_rect * xt           # multiplication by rectangular window
xwt_Hann = wt_Hann * xt           # multiplication by Hann window
# Fourier transforms of windowed signals
# obtained by convolution in frequency domain
# discretize convolution integral:
# multiply by 0.01 to correct for step size
Xwf_rect = np.convolve(Wf_rect, Xf, 'same') * 0.01            # rectangular
Xwf_Hann = np.convolve(Wf_Hann, Xf, 'same') * 0.01            # Hann

# plot windowed signals in time domain
plt.figure()
plt.subplot(2,2,1)
plt.plot(t, xt,  color=burgundy, linewidth=2, label='$x(t)$')
plt.plot(t, xwt_rect, color='g', linewidth=2, label='$x_{w,rect}(t)$')
plt.plot(t, xwt_Hann, color='b', linewidth=2, label='$x_{w,Hann}(t)$')
plt.xlabel('$t$ [s]'); plt.ylabel('$x(t), x_w(t)$')

# plot windowing 'consequences' in frequency domain
plt.subplot(2,2,2)
plt.plot(f, Xf,  color=burgundy, linewidth=2, label='$X(f)$')
plt.plot(f, Xwf_rect, color='g', linewidth=2, label='$X_{w,rect}(f)$')
plt.plot(f, Xwf_Hann, color='b', linewidth=2, label='$X_{w,Hann}(f)$')
plt.xlabel('$f$ [Hz]'); plt.ylabel('$X(f), X_w(f)$')
```

III

Discrete time

9

Sampling

Exercises

e|9.5 Aliasing

e.9-1

Sinusoidal signal

$$x(t) = 10\cos(4\pi t)$$

is sampled with $f_s = 5$ Hz.

When considering Fourier transform $X_s(f)$ of the sampled signal $x_s(t)$:

(a) At which frequency, starting from zero, for positive frequencies, do we encounter the first Dirac pulse?

(b) Does aliasing occur in this case? Explain your answer.

e.9-2

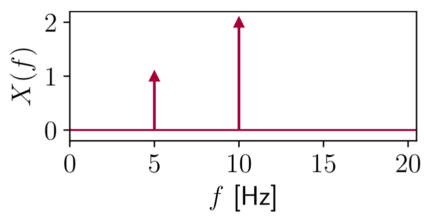
The positive frequencies of the spectrum of a continuous-time, real and even signal $x(t)$ are illustrated below on the right. The signal is sampled with $f_s = 20$ Hz.

Determine for each of the following statements whether the statement is true or false:

(I) The Nyquist rate equals 10 Hz.

(II) The Nyquist frequency equals 10 Hz.

(III) Aliasing does *not* occur.



e.9-3

Sinusoidal signal

$$x(t) = 2\cos(80\pi t)$$

is sampled with $f_s = 60$ Hz.

When considering Fourier transform $X_s(f)$ of the sampled signal $x_s(t)$, only for positive frequencies from 0 Hz to 175 Hz, *including* the 0 Hz and 175 Hz boundaries, at which frequencies do we encounter Dirac pulses?

e.9-4

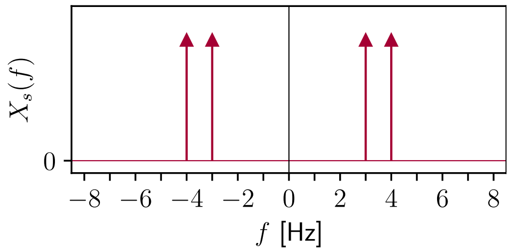
Sinusoidal signal

$$x(t) = 2\cos(8\pi t)$$

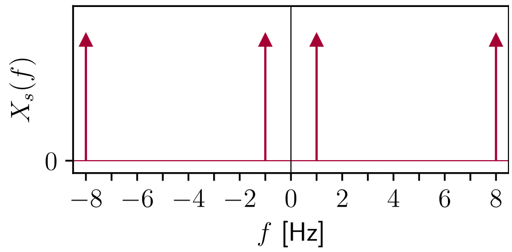
is sampled with $f_s = 7$ Hz.

Below four possible versions of the spectrum of the sampled signal $x_s(t)$, considering frequencies up to 8 Hz, are shown, of which only one is correct. Note that the weights of the Dirac pulses are deliberately not indicated. Which one is correct?

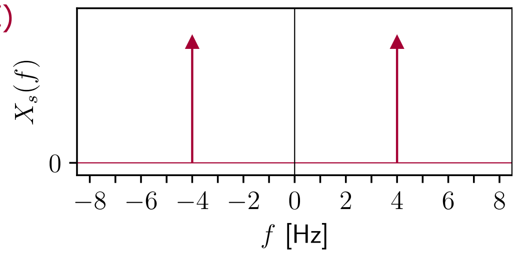
(I)



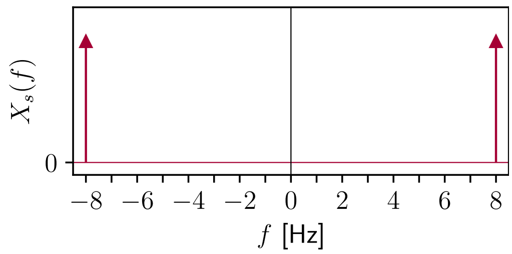
(III)



(II)



(IV)



e.9-5

Sinusoidal signal

$$x(t) = 3\cos(8\pi t) + 4\cos(34\pi t)$$

is sampled with $f_s = 10$ Hz.

We consider its Fourier transform $X(f)$, and also the Fourier transform of the sampled signal, $X_s(f)$, for positive frequencies only, from 0 Hz to 30 Hz, *including* the 0 Hz and 30 Hz boundaries.

Determine for each of the following statements whether the statement is true or false:

(I) The Nyquist rate equals 8 Hz.

(II) The Nyquist frequency equals 5 Hz.

(III) The first Dirac pulse of $X_s(f)$ in the specified frequency range appears at 4 Hz.

(IV) The Dirac pulse of $X_s(f)$ with the largest weight in the specified frequency range has a weight of 4.

e.9-6

Sinusoidal signal

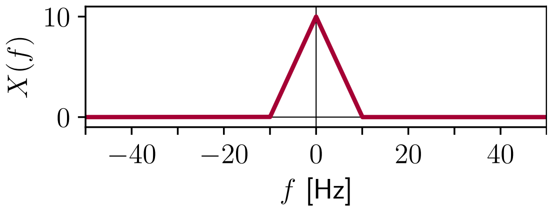
$$x(t) = 10\cos\left(10\pi t + \frac{\pi}{4}\right) + 4\cos\left(30\pi t - \frac{3\pi}{8}\right)$$

is sampled with $f_s = 11$ Hz.

When considering Fourier transform $X_s(f)$ of the sampled signal $x_s(t)$, only for positive frequencies from 0 Hz to 10 Hz, *including* the 0 Hz and 10 Hz boundaries, at which frequencies do we encounter Dirac pulses?

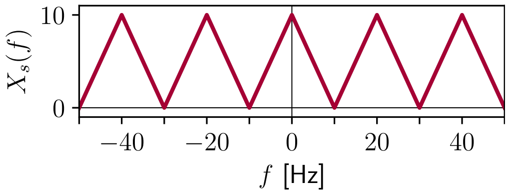
e.9-7

Signal $x(t)$ with Fourier transform $X(f)$ as shown here on the right is sampled with $f_s = 10$ Hz, yielding the sampled signal $x_s(t)$.

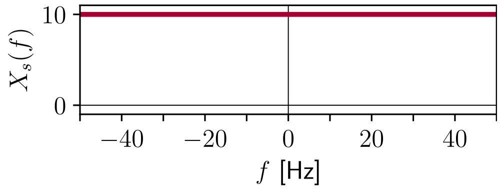


Below four possible versions of the spectrum of the sampled signal $x_s(t)$, considering frequencies up to 50 Hz, are shown, of which only one is correct. Which one is correct?

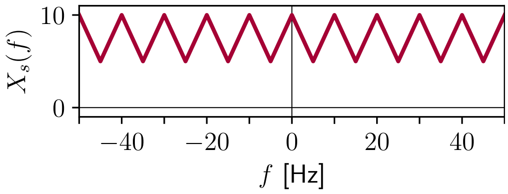
(I)



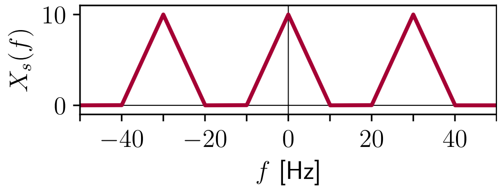
(III)



(II)



(IV)



e.9-8

Sinusoidal signal

$$x(t) = 7 + 14\cos(14\pi t)$$

is sampled with $f_s = 28$ Hz.

Draw an illustration similar to Figure 9.9 to show the Fourier transforms $X(f)$, $X_s(f)$ and $X_r(f)$ of, respectively, original signal $x(t)$, sampled signal $x_s(t)$ and reconstructed signal $x_r(t)$. Draw all components in the frequency band from $f = -100$ to 100 Hz. Use the band from $f = -\frac{f_s}{2}$ to $\frac{f_s}{2}$ Hz of $X_s(f)$ for $X_r(f)$.

e.9-9

Again consider sinusoidal signal

$$x(t) = 7 + 14\cos(14\pi t)$$

from Exercise e.9-8. This time, signal $x(t)$ is sampled with $f_s = 12$ Hz.

Again draw an illustration similar to Figure 9.9 to show the Fourier transforms $X(f)$, $X_s(f)$ and $X_r(f)$ of, respectively, original signal $x(t)$, sampled signal $x_s(t)$ and reconstructed signal $x_r(t)$. Draw all components in the frequency band from $f = -45$ to 45 Hz. Use the band from $f = -\frac{f_s}{2}$ to $\frac{f_s}{2}$ Hz of $X_s(f)$ for $X_r(f)$.

e.9-10

Sinusoidal signal

$$x(t) = 5 + 7\cos(8\pi t) + 8\cos(16\pi t) + 4\cos(20\pi t)$$

is sampled with $f_s = 30$ Hz.

Draw an illustration similar to Figure 9.9 to show the Fourier transforms $X(f)$, $X_s(f)$ and $X_r(f)$ of, respectively, original signal $x(t)$, sampled signal $x_s(t)$ and reconstructed signal $x_r(t)$. Draw all components in the frequency band from $f = -100$ to 100 Hz. Use the band from $f = -\frac{f_s}{2}$ to $\frac{f_s}{2}$ Hz of $X_s(f)$ for $X_r(f)$.

e.9-11 Signal

$$x(t) = 4\,\text{sinc}(6t)\cos^2(20\pi t)$$

needs to be sampled.

Which of the following statements is correct?

A. Signal $x(t)$ cannot be sampled.

B. Signal $x(t)$ needs to be sampled with at least 23 Hz in order to be able to reconstruct the original signal from its samples.

C. The Nyquist rate equals 40 Hz.

D. None of the other statements is correct.

e.9-12 Signal

$$x(t) = 2\,\text{sinc}^2(2t)\cos(20\pi t)$$

is sampled.

Which of the following statements is correct?

A. The Nyquist rate equals 22 Hz.

B. Signal $x(t)$ can be sampled with 24 Hz without aliasing to occur.

C. To avoid aliasing, signal $x(t)$ should be sampled with a sampling frequency that is strictly higher than 24 Hz.

D. None of the other statements is correct.

e.9-13 Signal

$$x(t) = 3\cos(80\pi t) + 2\cos(60\pi t)\,\text{sinc}(20t)$$

is sampled.

Which of the following statements is correct?

A. The Nyquist rate equals 40 Hz.

B. Signal $x(t)$ can be sampled with 80 Hz without aliasing to occur.

C. To avoid aliasing, signal $x(t)$ should be sampled with a sampling frequency that is strictly higher than 40 Hz.

D. None of the other statements is correct.

e.9-14 Signal

$$x(t) = 2\cos(4\pi t + \theta)$$

with arbitrary phase θ (in [rad]), is (intentionally) sampled at its Nyquist rate.

Show what happens in both the frequency domain *and* time domain, for:

(a) $\theta = 0$ (b) $\theta = \frac{\pi}{4}$ (c) $\theta = \frac{\pi}{2}$

e.9-15 Again consider sinusoidal signal

$$x(t) = 7 + 14\cos(14\pi t)$$

from Exercises e.9-8 and e.9-9. This time, signal $x(t)$ is sampled with $f_s = 14$ Hz.

Draw the Fourier transform of sampled signal $x_s(t)$, $X_s(f)$, in the frequency band from $f = -50$ to 50 Hz.

e.9-16

The sampling theorem, discussed in Section 9.4, applies to band-limited signals, meaning signals that have no frequency components beyond $f = f_h$: $X(f > f_h) = 0$. When f_h is known, we can select the proper sampling frequency f_s. In this exercise we will investigate what happens when sampling a non-band-limited signal.

Consider sampling a pulse

$$x(t) = A\Pi\left(\frac{t}{\tau}\right),$$

the Fourier transform of which equals

$$X(f) = A\tau\,\mathrm{sinc}(\tau f). \tag{5.11}$$

(a) What is the highest frequency f_h occurring in $X(f)$?

(b) Assume $A = 1$ and $\tau = 2$. Explain what happens when sampling $x(t)$ with $f_s = 2$ Hz.

(c) What could be a way to approximate the bandwidth of this signal? Explain the inevitable trade-off.

(d) Suppose we would like to be sure that 99% of the total energy of this signal is properly maintained after sampling. What would be the sampling frequency required?

Hint: recall the results from Exercise *e*.5-12.

Solutions

s|9.5 Aliasing

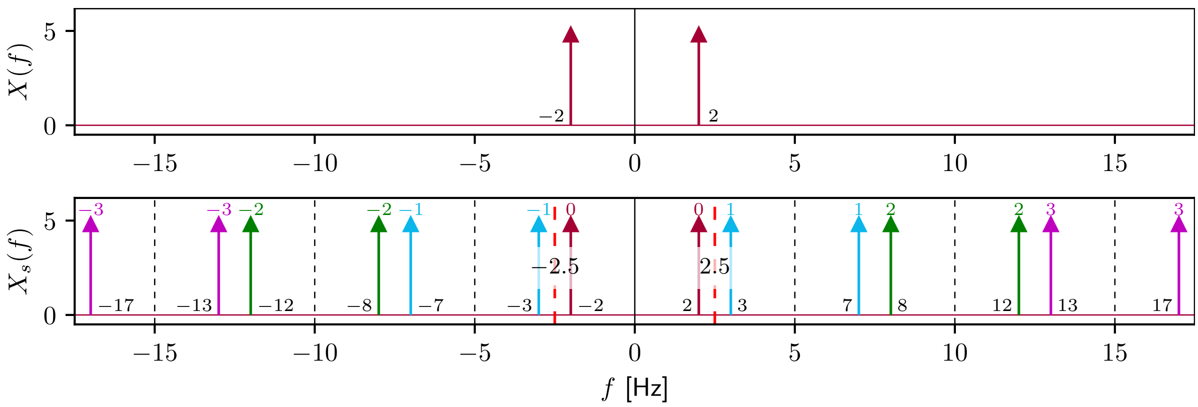
S.9-1

(a) The first Dirac pulse occurs at 2 Hz. See below for an elaborate explanation.

(b) Aliasing does *not* occur, since the sampling frequency $f_s = 5$ Hz, is *strictly larger* than $2f_h$, where f_h equals 2 Hz.

Following (5.16), signal $x(t) = 10\cos(4\pi t)$ with amplitude $A = 10$ and frequency $f_0 = 2$ Hz has Fourier transform in-the-limit $X(f) = \frac{A}{2}\big(\delta(f+f_0)+\delta(f-f_0)\big) = 5\big(\delta(f+2)+\delta(f-2)\big)$. The highest frequency occurring in $X(f)$, f_h, equals 2 Hz. Following the sampling theorem (9.12), since sampling frequency $f_s = 5$ Hz is *strictly larger* than $2f_h$, aliasing does not occur. As a result, the first Dirac pulse (for positive frequencies) occurs at the (positive) frequency of the Dirac pulse of Fourier transform $X(f)$ of original signal $x(t)$, hence, at 2 Hz.

The figure below shows the Fourier transform of the original signal $X(f)$ (top) and that of the sampled signal $X_s(f)$ (bottom). The first copy of $X(f)$ is positioned at $f = 0f_s$ (for $k = 0$), two Dirac pulses at ± 2 Hz (burgundy). One copy is positioned at $f = 1f_s$ ($k = 1$), yielding Dirac pulses at $5-2 = 3$ Hz (blue) and $5+2 = 7$ Hz (blue). Another copy is positioned at $f = -1f_s$ ($k = -1$), yielding Dirac pulses at $-5-2 = -7$ Hz (blue) and $-5+2 = -3$ Hz (blue). The 'next' copies appear at $f = 2f_s$ ($k = 2$), yielding Dirac pulses at $10-2 = 8$ Hz (green) and $10+2 = 12$ Hz (green), and at $f = -2f_s$ ($k = -2$), yielding Dirac pulses at $-10-2 = -12$ Hz (green) and $-10+2 = 8$ Hz (green). Again, the 'next' copies are placed at at $f = 3f_s$ ($k = 3$), yielding Dirac pulses at $15-2 = 13$ Hz (magenta) and $15+2 = 17$ Hz (magenta), and at $f = -3f_s$ ($k = -3$), yielding Dirac pulses at $-15-2 = -17$ Hz (magenta) and $-15+2 = -13$ Hz (magenta). All other copies are outside the range of the shown frequency axis.



S.9-2

(I) False, as the Nyquist rate equals $2f_h$, with f_h being the largest frequency occurring in Fourier transform $X(f)$ of signal $x(t)$. In this case, the Nyquist rate thus equals $2\cdot 10 = 20$ Hz.

(II) True, as the Nyquist frequency equals $\frac{f_s}{2}$, with f_s being the sampling frequency. In this case, $f_s = 20$ Hz, and thus the Nyquist frequency equals $\frac{20}{2} = 10$ Hz.

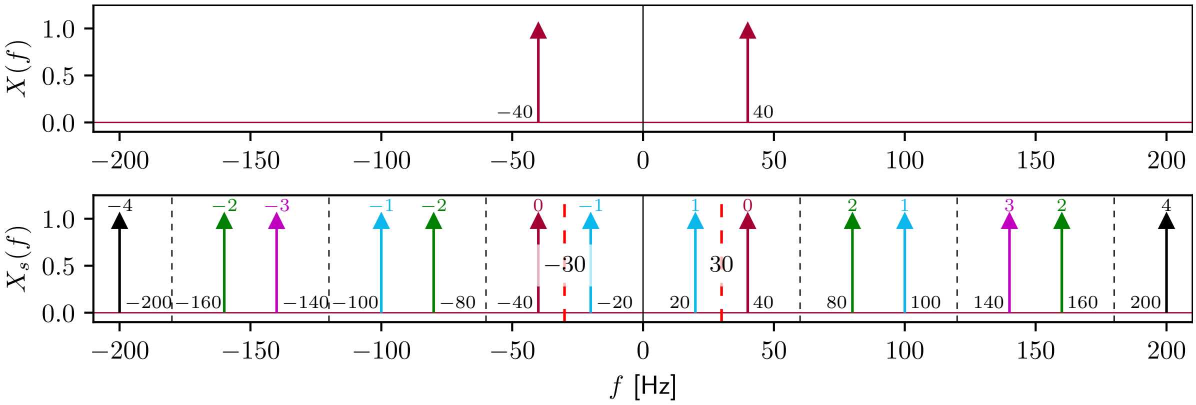
(III) False, as the highest frequency occurring in $X(f)$, f_h, equals 10 Hz, and at $f_h = 10$ Hz, $X(f = f_h)$ is non-zero. Therefore, for aliasing not to occur the sampling frequency f_s should be strictly larger than $2f_h$, but in this case it equals $2f_h$.

S.9-3

Following (5.16), signal $x(t) = 2\cos(80\pi t)$ with frequency $f_0 = 40$ Hz has Fourier transform $X(f) = \delta(f+40) + \delta(f-40)$. The highest frequency occurring in $X(f)$, f_h, equals 40 Hz. Since sampling frequency $f_s = 60$ Hz, is *smaller* than $2f_h$, aliasing *does* occur.

The figure at the bottom of this solution shows the Fourier transform of the original signal $X(f)$ (top) and that of the sampled signal $X_s(f)$ (bottom). The table below shows the frequencies at which Dirac pulses are positioned in $X_s(f)$. Since the Dirac pulses for $k = 4$ already end up outside the frequency range of interest, and those for $k = -1$ appear on the negative frequency axis, the Dirac pulses of interest originate from $k = 0, 1, 2, 3$. There are six Dirac pulses between 0 Hz and 175 Hz at the following frequencies (highlighted in gray): 20 Hz, 40 Hz, 80 Hz, 100 Hz, 140 Hz and 160 Hz.

k	kf_s	kf_s [Hz]	$f_{\text{Dirac\_1}}$ [Hz]	$f_{\text{Dirac\_2}}$ [Hz]
0	$0f_s$	0	$0-40 = -40$	$0+40 = 40$
1	$1f_s$	60	$60-40 = 20$	$60+40 = 100$
2	$2f_s$	120	$120-40 = 80$	$120+40 = 160$
3	$3f_s$	180	$180-40 = 140$	$180+40 = 220$
4	$4f_s$	240	$240-40 = 200$	$240+40 = 280$
-1	$-1f_s$	-60	$-60-40 = -100$	$-60+40 = -20$



S.9-4

The correct spectrum is spectrum (I).

Following (5.16), signal $x(t) = 2\cos(8\pi t)$ with frequency $f_0 = 4$ Hz has Fourier transform $X(f) = \delta(f+4) + \delta(f-4)$. The highest frequency occurring in $X(f)$, f_h, equals 4 Hz. Since sampling frequency $f_s = 7$ Hz is *smaller* than $2f_h$, aliasing *does* occur.

The table below shows the frequencies at which Dirac pulses occur in $X_s(f)$. Since the Dirac pulses for $k = -2$ and $k = 2$ already end up outside the frequency range of interest, the Dirac pulses of interest originate from $k = -1, 0, 1$. There are four Dirac pulses between -8 and 8 Hz at the following frequencies (highlighted in gray): -4 Hz, -3 Hz, 3 Hz and 4 Hz.

k	kf_s	kf_s [Hz]	$f_{\text{Dirac\_1}}$ [Hz]	$f_{\text{Dirac\_2}}$ [Hz]
0	$0f_s$	0	$0-4 = -4$	$0+4 = 4$
1	$1f_s$	7	$7-4 = 3$	$7+4 = 11$
2	$2f_s$	14	$14-4 = 10$	$14+4 = 18$
-1	$-1f_s$	-7	$-7-4 = -11$	$-7+4 = -3$
-2	$-2f_s$	-14	$-14-4 = -18$	$-14+4 = -10$

S.9-5

(I) False, the Nyquist rate equals 34 Hz (see below).

(II) True, the Nyquist frequency equals $\frac{f_s}{2} = \frac{10}{2} = 5$ Hz.

(III) False, the first Dirac pulse appears at 3 Hz (see below).

(IV) False, the Dirac pulse with the largest weight has a weight of 2 (see below).

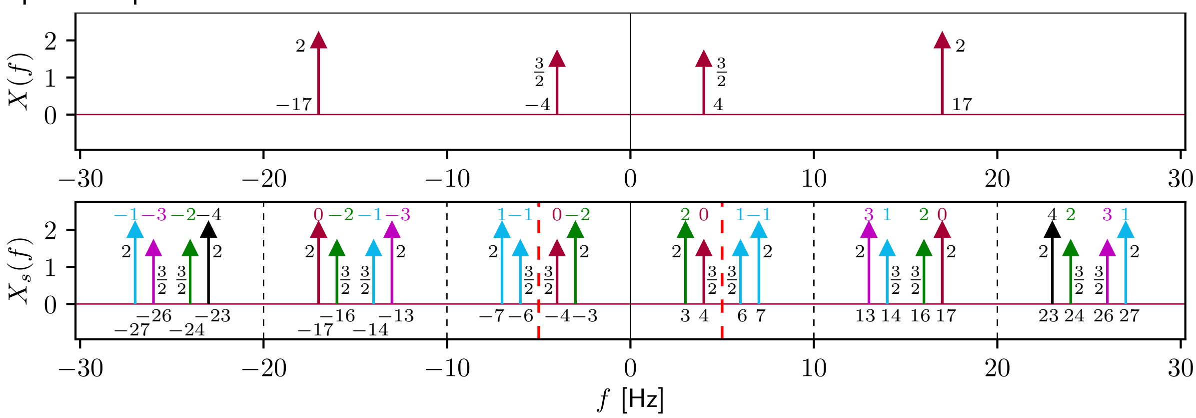
Signal $x(t) = 3\cos(8\pi t) + 4\cos(34\pi t)$ has two components with frequencies $f_1 = 4$ Hz and $f_2 = 17$ Hz, respectively. Following (5.16), its Fourier transform is $X(f) = \frac{3}{2}\big(\delta(f+4) + \delta(f-4)\big) + 2\big(\delta(f+17) + \delta(f-17)\big)$. The highest frequency occurring in $X(f)$, f_h, equals 17 Hz. At this frequency, we have a Dirac pulse, i.e., $X(f = f_h) \neq 0$. Therefore, for aliasing not to occur, we must apply a sampling frequency that is strictly larger than $2f_h$, i.e., $f_s > 34$ Hz. Since we sample at 10 Hz, aliasing *does* occur. The signal's Nyquist rate equals $2f_h = 2 \cdot 17 = 34$ Hz. Hence, statement (I) is incorrect.

The two figures at the bottom of this solution both show the Fourier transform of the original signal $X(f)$ (top) and that of the sampled signal $X_s(f)$ (bottom). Although the two figures show the same Fourier transforms, the Dirac pulses are drawn slightly differently to illustrate that both representations are acceptable. Note not only the differences in the lengths of the Dirac pulses, but also the differences in the values shown on the vertical axis. In the remainder of this book, only one of the two forms of spectral representation is used in exercises and solutions when illustrating Fourier transforms. Typically, the top representation is used, although the bottom one is sometimes preferred when it offers better visibility.

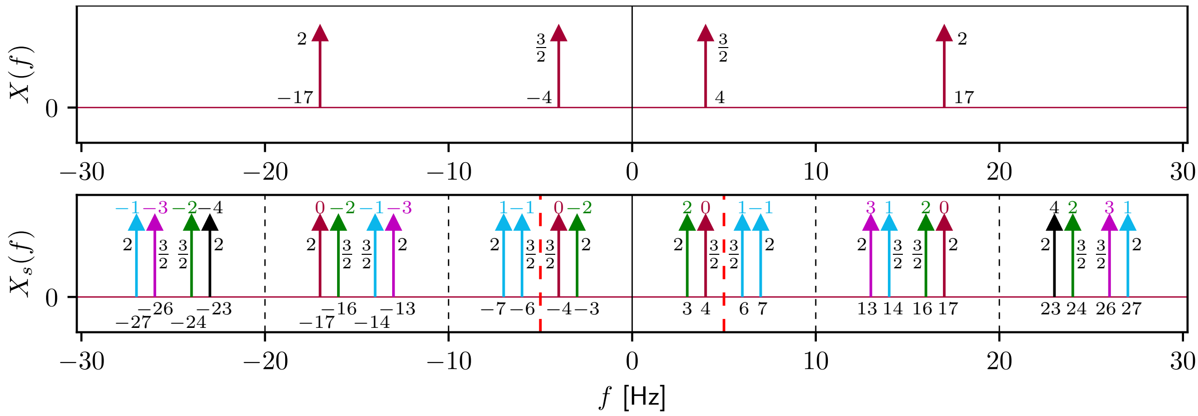
The table below shows the frequencies at which Dirac pulses occur in $X_s(f)$. Since the Dirac pulses for $k = 5$ already end up outside the frequency range of interest, and those for $k = -2$ all appear on the negative frequency axis, the Dirac pulses of interest originate from $k = -1, 0, 1, 2, 3, 4$. The first Dirac pulse of $X_s(f)$ in the frequency range of interest appears at 3 Hz. Hence, statement (III) is incorrect. In the frequency range of interest, the Dirac pulse of $X_s(f)$ with the largest weight has a weight of 2. Hence, statement (IV) is also incorrect.

k	kf_s	kf_s [Hz]	$f_{\text{Dirac\_1}}$ [Hz]	$f_{\text{Dirac\_2}}$ [Hz]	$f_{\text{Dirac\_3}}$ [Hz]	$f_{\text{Dirac\_4}}$ [Hz]
0	$0f_s$	0	$0-17 = -17$	$0-4 = -4$	$0+4 = 4$	$0+17 = 17$
1	$1f_s$	10	$10-17 = -7$	$10-4 = 6$	$10+4 = 14$	$10+17 = 27$
2	$2f_s$	20	$20-17 = 3$	$20-4 = 16$	$20+4 = 24$	$20+17 = 37$
3	$3f_s$	30	$30-17 = 13$	$30-4 = 26$	$30+4 = 34$	$30+17 = 47$
4	$4f_s$	40	$40-17 = 23$	$40-4 = 36$	$40+4 = 44$	$40+17 = 57$
5	$5f_s$	50	$50-17 = 33$	$50-4 = 46$	$50+4 = 54$	$50+17 = 67$
-1	$-1f_s$	-10	$-10-17 = -27$	$-10-4 = -14$	$-10+4 = -6$	$-10+17 = 7$
-2	$-2f_s$	-20	$-20-17 = -37$	$-20-4 = -24$	$-20+4 = -16$	$-20+17 = -3$

Spectral representation 1:



Spectral representation 2:

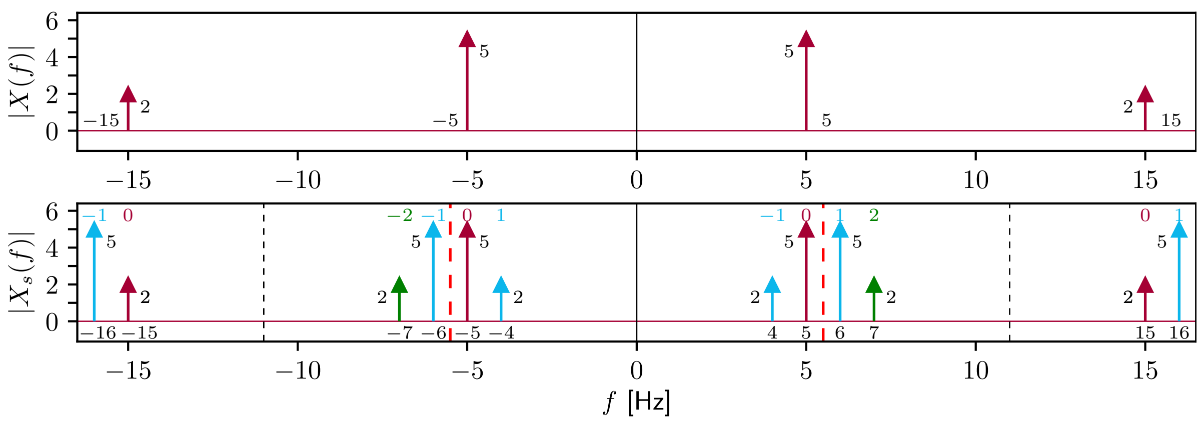


S.9-6

Sinusoidal signal $x(t) = 10\cos\left(10\pi t + \frac{\pi}{4}\right) + 4\cos\left(30\pi t - \frac{3\pi}{8}\right)$ has two components with frequencies $f_1 = 5$ Hz and $f_2 = 15$ Hz, respectively. Hence, the highest frequency occurring in $X(f)$, f_h, equals 15 Hz, and at this frequency, we have a Dirac pulse, i.e., $X(f = f_h) \neq 0$. The signal's Nyquist rate equals $2f_h = 2 \cdot 15 = 30$ Hz. For aliasing not to occur, we must apply a sampling frequency that is strictly larger than $2f_h$, i.e., $f_s > 30$ Hz. However, since we sample at 11 Hz, aliasing *does* occur.

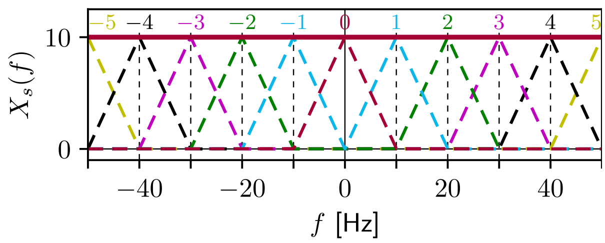
The figure at the bottom of this solution shows the amplitude spectrum of the original signal $|X(f)|$ (top) and that of the sampled signal $|X_s(f)|$ (bottom). The table below shows the frequencies at which Dirac pulses occur in $X_s(f)$. Since the Dirac pulses for $k = 3$ already end up outside the frequency range of interest, and those for $k = -2$ all appear on the negative frequency axis, the Dirac pulses of interest originate from $k = -1, 0, 1, 2$. There are four Dirac pulses between 0 Hz and 10 Hz at the following frequencies (highlighted in gray): 4 Hz, 5 Hz, 6 Hz and 7 Hz.

k	kf_s	kf_s [Hz]	$f_{\text{Dirac\_1}}$ [Hz]	$f_{\text{Dirac\_2}}$ [Hz]	$f_{\text{Dirac\_3}}$ [Hz]	$f_{\text{Dirac\_4}}$ [Hz]
0	$0f_s$	0	$0 - 15 = -15$	$0 - 5 = -5$	$0 + 5 = 5$	$0 + 15 = 15$
1	$1f_s$	11	$11 - 15 = -4$	$11 - 5 = 6$	$11 + 5 = 16$	$11 + 15 = 26$
2	$2f_s$	22	$22 - 15 = 7$	$22 - 5 = 17$	$22 + 5 = 27$	$22 + 15 = 37$
3	$3f_s$	33	$33 - 15 = 18$	$33 - 5 = 28$	$33 + 5 = 38$	$33 + 15 = 48$
-1	$-1f_s$	-11	$-11 - 15 = -26$	$-11 - 5 = -16$	$-11 + 5 = -6$	$-11 + 15 = 4$
-2	$-2f_s$	-22	$-22 - 15 = -37$	$-22 - 5 = -27$	$-22 + 5 = -17$	$-22 + 15 = -7$



S.9-7

The correct spectrum is spectrum (III).



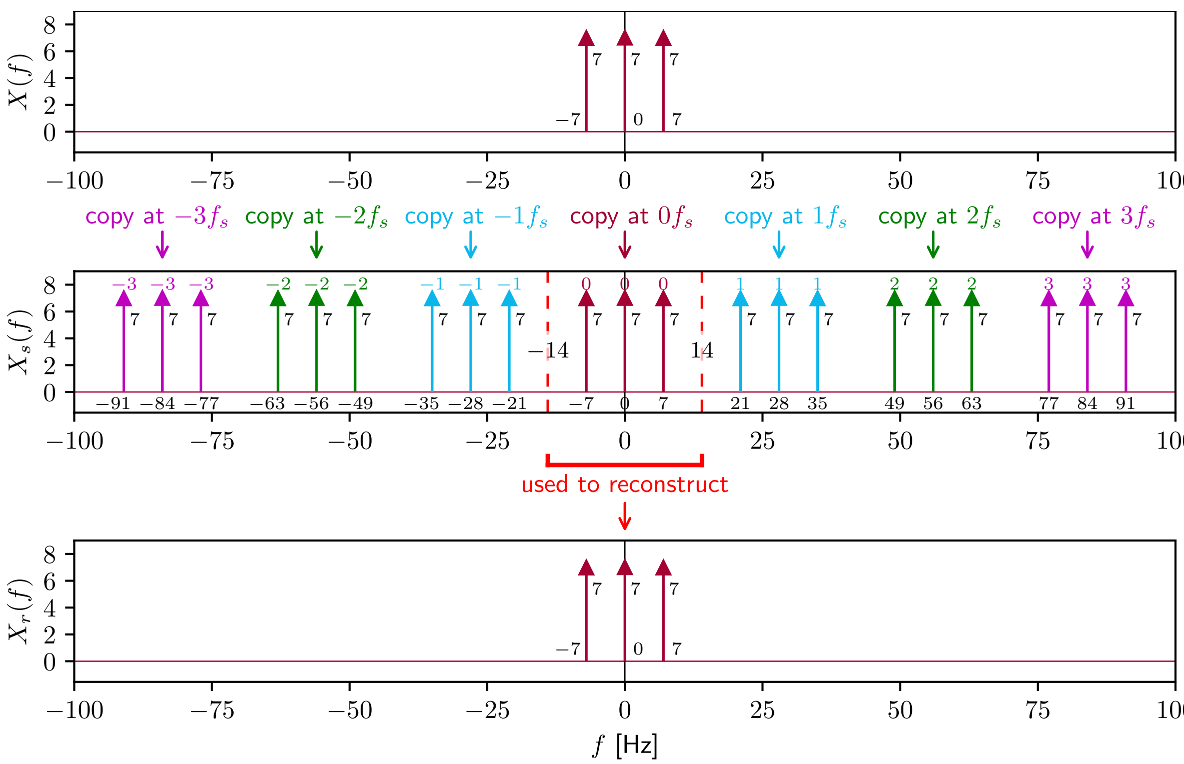
The highest frequency occurring in $X(f)$, f_h, equals 10 Hz. Since sampling frequency $f_s = 10$ Hz equals f_h, and is thus smaller than $2f_h$, aliasing *does* occur.

The figure on the right shows the Fourier transform of the sampled signal $X_s(f)$ (solid burgundy), as well as the individual copies (dashed individual colors) of the original $X(f)$, occurring at all integer multiples of the sampling frequency f_s, i.e., the contributions that together form Fourier transform $X_s(f)$, cf. (9.10). Since $f_s = f_h$, the copies overlap such that the center of one copy aligns with the edge frequencies of adjacent copies. At all frequencies, the total contribution of the copies is constant and equal to 10, resulting in a flat (horizontal) spectrum.

S.9-8

Sinusoidal signal $x(t) = 7 + 14\cos(14\pi t)$ has two components, with frequencies $f_1 = 0$ Hz and $f_2 = 7$ Hz, respectively. Following (5.15) and (5.16), its Fourier transform is $X(f) = 7\delta(f) + 7\delta(f+7) + 7\delta(f-7)$. The highest frequency occurring in $X(f)$, f_h, equals 7 Hz, and at this frequency, we have a Dirac pulse, i.e., $X(f = f_h) \neq 0$. The signal's Nyquist rate equals $2f_h = 2 \cdot 7 = 14$ Hz. Since sampling frequency f_s equals 28 Hz, the sampling system's Nyquist frequency equals $\frac{28}{2} = 14$ Hz. Since f_s is strictly larger than $2f_h$, aliasing does *not* occur and original signal $x(t)$ can be reconstructed.

The figure below shows the Fourier transform of the original signal $X(f)$ (top), that of the sampled signal $X_s(f)$ (middle) and that of the reconstructed signal $X_r(f)$ (bottom), where we have used $X_s(f)$ between $\pm\frac{f_s}{2}$ Hz to reconstruct the signal. $X_r(f)$ is *identical* to $X(f)$, as it should be, since we already determined above that aliasing does not occur. The table at the bottom of this solution shows the frequencies at which Dirac pulses occur in $X_s(f)$.

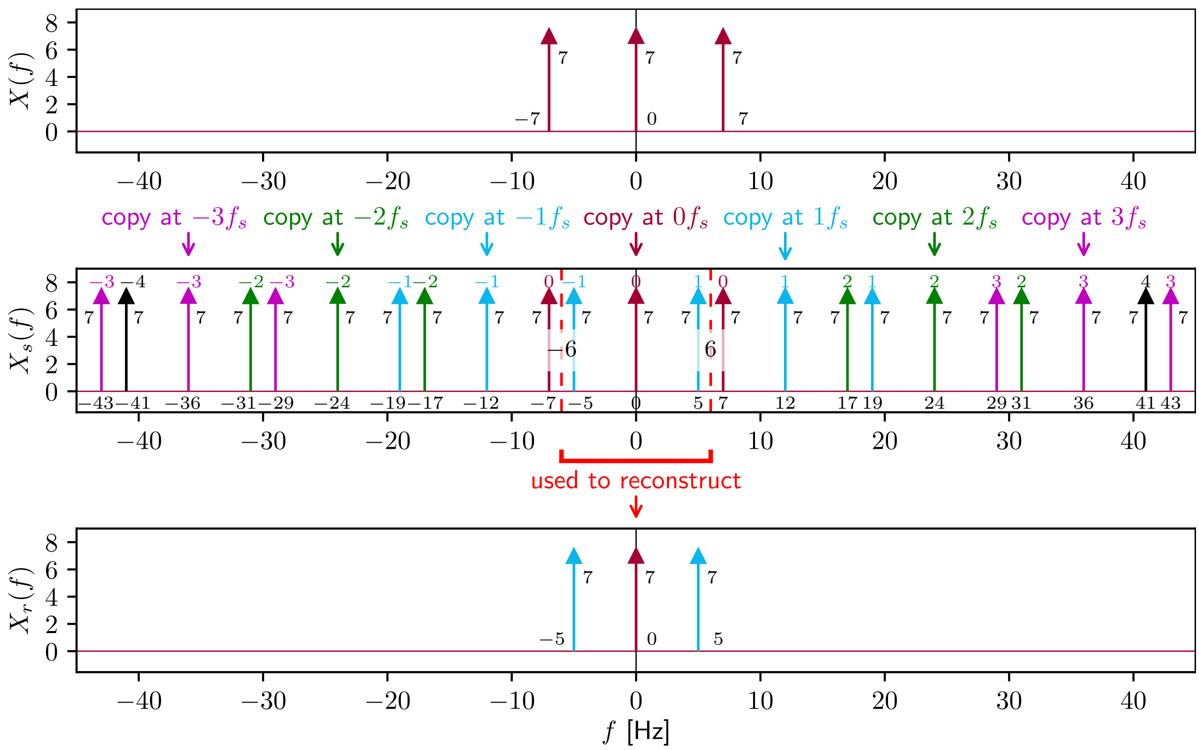


k	kf_s	kf_s [Hz]	$f_{\text{Dirac\_1}}$ [Hz]	$f_{\text{Dirac\_2}}$ [Hz]	$f_{\text{Dirac\_3}}$ [Hz]
0	$0f_s$	0	$0-7 = -7$	$0+0 = 0$	$0+7 = 7$
1	$1f_s$	28	$28-7 = 21$	$28+0 = 28$	$28+7 = 35$
2	$2f_s$	56	$56-7 = 49$	$56+0 = 56$	$56+7 = 63$
3	$3f_s$	84	$84-7 = 77$	$84+0 = 84$	$84+7 = 91$
4	$4f_s$	112	$112-7 = 105$	$112+0 = 112$	$112+7 = 119$
−1	$-1f_s$	−28	$-28-7 = -35$	$-28+0 = -28$	$-28+7 = -21$
−2	$-2f_s$	−56	$-56-7 = -63$	$-56+0 = -56$	$-56+7 = -49$
−3	$-3f_s$	−84	$-84-7 = -91$	$-84+0 = -84$	$-84+7 = -77$
−4	$-4f_s$	−112	$-112-7 = -119$	$-112+0 = -112$	$-112+7 = -105$

S.9-9

As discussed in Solution *S*.9-8, sinusoidal signal $x(t) = 7 + 14\cos(14\pi t)$ has two components, with frequencies $f_1 = 0$ Hz and $f_2 = 7$ Hz, respectively. Hence, the highest frequency in $X(f)$, f_h, equals 7 Hz. According to the sampling theorem (9.12), sampling frequency $f_s = 12$ Hz is too low, as $f_s < 2f_h$, and aliasing *does* occur. Original signal $x(t)$ *cannot* be reconstructed.

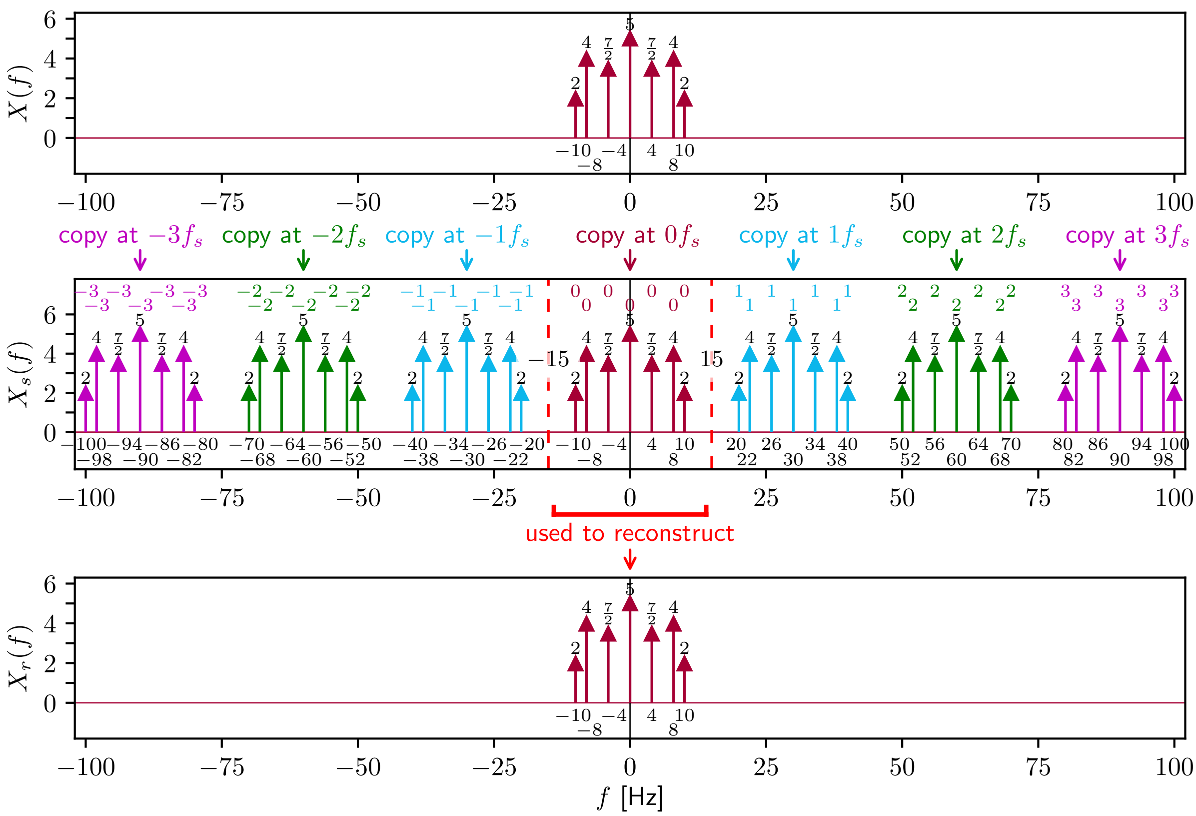
The figure below shows the Fourier transform of the original signal $X(f)$ (top; the same as in Solution *S*.9-8), that of the sampled signal $X_s(f)$ (middle) and that of the reconstructed signal $X_r(f)$ (bottom), where we have used $X_s(f)$ between $\pm\frac{f_s}{2}$ Hz to reconstruct the signal. Indeed, as already deduced above, $X_r(f)$ is clearly different from $X(f)$. Reconstructed signal $x_r(t)$ has a component with a frequency of $f_2 = 5$ Hz instead of the original signal's $f_2 = 7$ Hz. With the current sampling frequency, the original signal cannot be reconstructed from the sampled signal, independent of the chosen frequency band used for reconstruction. The spectrum of original signal $x(t)$ overlaps with the spectra of the signal's aliases, i.e., aliasing occurs. The energy coming from the aliases cannot be distinguished from the energy of the original signal. Only a larger sampling frequency can help to reconstruct original signal $x(t)$. The table at the bottom of this solution shows the frequencies at which Dirac pulses occur in $X_s(f)$.



k	kf_s	kf_s [Hz]	$f_{\text{Dirac\_1}}$ [Hz]	$f_{\text{Dirac\_2}}$ [Hz]	$f_{\text{Dirac\_3}}$ [Hz]
0	$0f_s$	0	$0-7 = -7$	$0+0 = 0$	$0+7 = 7$
1	$1f_s$	12	$12-7 = 5$	$12+0 = 12$	$12+7 = 19$
2	$2f_s$	24	$24-7 = 17$	$24+0 = 24$	$24+7 = 31$
3	$3f_s$	36	$36-7 = 29$	$36+0 = 36$	$36+7 = 43$
4	$4f_s$	48	$48-7 = 41$	$48+0 = 48$	$48+7 = 55$
-1	$-1f_s$	-12	$-12-7 = -19$	$-12+0 = -12$	$-12+7 = -5$
-2	$-2f_s$	-24	$-24-7 = -31$	$-24+0 = -24$	$-24+7 = -17$
-3	$-3f_s$	-36	$-36-7 = -43$	$-36+0 = -36$	$-36+7 = -29$
-4	$-4f_s$	-48	$-48-7 = -55$	$-48+0 = -48$	$-48+7 = -41$

S.9-10 Sinusoidal signal $x(t) = 5 + 7\cos(8\pi t) + 8\cos(16\pi t) + 4\cos(20\pi t)$ has four components, with frequencies $f_1 = 0$ Hz, $f_2 = 4$ Hz, $f_3 = 8$ Hz and $f_4 = 10$ Hz, respectively. Following (5.15) and (5.16), $x(t)$ its Fourier transform is $X(f) = 5\delta(f) + \frac{7}{2}\delta(f+4) + \frac{7}{2}\delta(f-4)$ $+ 4\delta(f+8) + 4\delta(f-8) + 2\delta(f+10) + 2\delta(f-10)$. Hence, the highest frequency occurring in $X(f)$, f_h, equals 10 Hz, and at this frequency, we have a Dirac pulse, i.e., $X(f = f_h) \neq 0$. The signal's Nyquist rate equals $2f_h = 2 \cdot 10 = 20$ Hz. Since sampling frequency f_s equals 30 Hz, the sampling system's Nyquist frequency equals $\frac{30}{2} = 15$ Hz. Since f_s is strictly larger than $2f_h$, aliasing does *not* occur and original signal $x(t)$ can be reconstructed.

The figure below shows the Fourier transform of the original signal $X(f)$ (top), that of the sampled signal $X_s(f)$ (middle) and that of the reconstructed signal $X_r(f)$ (bottom), where we have used $X_s(f)$ between $\pm\frac{f_s}{2}$ Hz to reconstruct the signal. $X_r(f)$ is *identical* to $X(f)$, as it should be, since we already determined above that aliasing does not occur. The table at the bottom of this solution shows the frequencies at which Dirac pulses occur in $X_s(f)$.



k	kf_s	kf_s [Hz]	$f_{\text{Dirac\_1}}$ [Hz]	$f_{\text{Dirac\_2}}$ [Hz]	$f_{\text{Dirac\_3}}$ [Hz]	$f_{\text{Dirac\_4}}$ [Hz]
0	$0f_s$	0	$0-10=-10$	$0-8=-8$	$0-4=-4$	$0+0=0$
1	$1f_s$	30	$30-10=20$	$30-8=22$	$30-4=26$	$30+0=30$
2	$2f_s$	60	$60-10=50$	$60-8=52$	$60-4=56$	$60+0=60$
3	$3f_s$	90	$90-10=80$	$90-8=82$	$90-4=86$	$90+0=90$
4	$4f_s$	120	$120-10=110$	$120-8=112$	$120-4=116$	$120+0=120$
-1	$-1f_s$	-30	$-30-10=-40$	$-30-8=-38$	$-30-4=-34$	$-30+0=-30$
-2	$-2f_s$	-60	$-60-10=-70$	$-60-8=-68$	$-60-4=-64$	$-60+0=-60$
-3	$-3f_s$	-90	$-90-10=-100$	$-90-8=-98$	$-90-4=-94$	$-90+0=-90$
-4	$-4f_s$	-120	$-120-10=-130$	$-120-8=-128$	$-120-4=-124$	$-120+0=-120$

k	kf_s	kf_s [Hz]	$f_{\text{Dirac\_5}}$ [Hz]	$f_{\text{Dirac\_6}}$ [Hz]	$f_{\text{Dirac\_7}}$ [Hz]
0	$0f_s$	0	$0+4=4$	$0+8=8$	$0+10=10$
1	$1f_s$	30	$30+4=34$	$30+8=38$	$30+10=40$
2	$2f_s$	60	$60+4=64$	$60+8=68$	$60+10=70$
3	$3f_s$	90	$90+4=94$	$90+8=98$	$90+10=100$
4	$4f_s$	120	$120+4=124$	$120+8=128$	$120+10=130$
-1	$-1f_s$	-30	$-30+4=-26$	$-30+8=-22$	$-30+10=-20$
-2	$-2f_s$	-60	$-60+4=-56$	$-60+8=-52$	$-60+10=-50$
-3	$-3f_s$	-90	$-90+4=-86$	$-90+8=-82$	$-90+10=-80$
-4	$-4f_s$	-120	$-120+4=-116$	$-120+8=-112$	$-120+10=-110$

S.9-11

Statement D. is correct.

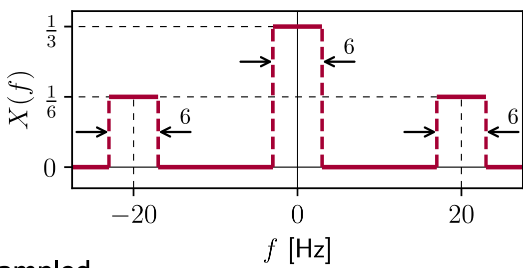
Using $\cos^2 u = \frac{1}{2}(1+\cos(2u))$, (A.4):

$$x(t) = 4\,\text{sinc}(6t)\cos^2(20\pi t) = 4\,\text{sinc}(6t)\,\tfrac{1}{2}\big(1+\cos(40\pi t)\big)$$
$$= 2\,\text{sinc}(6t) + 2\,\text{sinc}(6t)\cos(40\pi t)$$

From $\Pi\left(\frac{t}{\tau}\right) \overset{\mathcal{F}}{\leftrightarrow} \tau\,\text{sinc}(\tau f)$, (5.11), and the duality theorem $X(t) \overset{\mathcal{F}}{\leftrightarrow} x(-f)$, (6.5), we know that: $\mathcal{F}\{\tau\,\text{sinc}(\tau t)\} = \Pi\left(\frac{-f}{\tau}\right) = \Pi\left(\frac{f}{\tau}\right)$, (6.13). Combining this with the modulation theorem (6.7), $x_1(t)\cos(2\pi f_0 t) \overset{\mathcal{F}}{\leftrightarrow} \frac{1}{2}X_1(f+f_0) + \frac{1}{2}X_1(f-f_0)$, we obtain for Fourier transform $X(f)$:

$$X(f) = 2\tfrac{1}{6}\Pi\left(\tfrac{f}{6}\right) + 2\tfrac{1}{2}\tfrac{1}{6}\Pi\left(\tfrac{f+20}{6}\right) + 2\tfrac{1}{2}\tfrac{1}{6}\Pi\left(\tfrac{f-20}{6}\right)$$
$$= \tfrac{1}{3}\Pi\left(\tfrac{f}{6}\right) + \tfrac{1}{6}\Pi\left(\tfrac{f+20}{6}\right) + \tfrac{1}{6}\Pi\left(\tfrac{f-20}{6}\right)$$

Fourier transform $X(f)$ is illustrated on the right. The highest frequency occurring in $X(f)$, f_h, equals 23 Hz. Hence, the signal's Nyquist rate equals $2f_h = 2\cdot 23 = 46$ Hz. Statement C. is incorrect. Looking at $X(f)$, this signal can be sampled without aliasing with frequencies equal to or larger than 46 Hz. Statement B. is incorrect.



Statement A. is nonsense; a signal can always be sampled.

S.9-12 Statement B. is correct.

Signal $x(t) = 2\,\text{sinc}^2(2t)\cos(20\pi t)$ is a multiplication of a sinc-squared function and a cosine component:

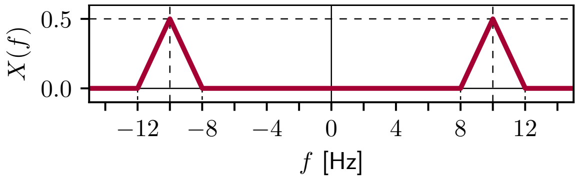
$$x(t) = a(t)\,b(t) \quad \text{with} \quad a(t) = 2\,\text{sinc}^2(2t) \quad \text{and} \quad b(t) = \cos(20\pi t)$$

Using $\tau\,\text{sinc}^2(\tau t) \overset{\mathcal{F}}{\leftrightarrow} \Lambda\left(\frac{f}{\tau}\right)$, (7.7), we obtain for Fourier transform $A(f)$: $\quad A(f) = \Lambda\left(\frac{f}{2}\right)$

Using the modulation theorem (6.7), $x_1(t)\cos(2\pi f_0 t) \overset{\mathcal{F}}{\leftrightarrow} \frac{1}{2}X_1(f+f_0) + \frac{1}{2}X_1(f-f_0)$, with $b(t) = \cos(20\pi t)$:

$$X(f) = \tfrac{1}{2}\Lambda\left(\frac{f+10}{2}\right) + \tfrac{1}{2}\Lambda\left(\frac{f-10}{2}\right)$$

The figure in the bottom right corner shows the Fourier transform of the original signal $X(f)$. The triangular function $\Lambda\left(\frac{f}{\tau}\right)$ has a width of 2τ. The highest frequency occurring in $X(f)$, f_h, thus equals 12 Hz. Hence, the Nyquist rate of signal $x(t)$ equals $2f_h = 2 \cdot 12 = 24$ Hz. Statement A. is incorrect. At $f_h = 12$ Hz, $X(f = f_h) = 0$, meaning that we can sample at the Nyquist rate of 24 Hz without aliasing to occur. Hence, statement B. is correct and statements C. and D. are incorrect.



S.9-13 Statement D. is correct.

Signal $\quad x(t) = 3\cos(80\pi t) + 2\cos(60\pi t)\,\text{sinc}(20t) = b(t) + c(t)d(t)$

with $\quad b(t) = 3\cos(80\pi t), \quad c(t) = 2\cos(60\pi t) \quad$ and $\quad d(t) = \text{sinc}(20t)$

Following (5.16), $b(t)$ with amplitude $A = 3$ and frequency $f_0 = 40$ Hz has Fourier transform:

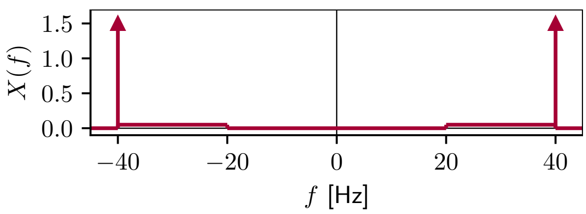
$$B(f) = \tfrac{A}{2}\big(\delta(f+f_0) + \delta(f-f_0)\big) = \tfrac{3}{2}\big(\delta(f+40) + \delta(f-40)\big)$$

From $\Pi\left(\frac{t}{\tau}\right) \overset{\mathcal{F}}{\leftrightarrow} \tau\,\text{sinc}(\tau f)$, (5.11), and the duality theorem $X(t) \overset{\mathcal{F}}{\leftrightarrow} x(-f)$, (6.5), we know that: $\mathcal{F}\{\tau\,\text{sinc}(\tau t)\} = \Pi\left(\frac{-f}{\tau}\right) = \Pi\left(\frac{f}{\tau}\right)$, (6.13). Hence: $\quad D(f) = \frac{1}{20}\Pi\left(\frac{f}{20}\right)$

Using the modulation theorem (6.7), $x_1(t)\cos(2\pi f_0 t) \overset{\mathcal{F}}{\leftrightarrow} \frac{1}{2}X_1(f+f_0) + \frac{1}{2}X_1(f-f_0)$, for Fourier-transforming $c(t)d(t)$, we obtain for $X(f)$:

$$X(f) = \tfrac{3}{2}\big(\delta(f+40) + \delta(f-40)\big) + \tfrac{1}{20}\left(\Pi\left(\frac{f+30}{20}\right) + \Pi\left(\frac{f-30}{20}\right)\right)$$

The figure in the bottom right corner shows the Fourier transform of the original signal$X(f)$. Two Dirac pulses occur at, respectively, $f = -40$ Hz and $f = 40$ Hz. Also, two pulse functions with a pulse width of 20 Hz centered at $f = -30$ Hz and $f = 30$ Hz, respectively, can be seen. The highest frequency in $X(f)$, f_h, equals 40 Hz. Hence, the Nyquist rate of signal $x(t)$ equals $2f_h = 2 \cdot 40 = 80$ Hz. Statement A. is incorrect. At $f_h = 40$ Hz, $X(f = f_h)$ is non-zero, as a Dirac pulse occurs at $f = 40$ Hz. Hence, signal $x(t)$ can only be sampled without aliasing to occur at sampling frequencies that are strictly larger than $2f_h = 80$ Hz. Hence, statements B. and C. are incorrect and statement D. is correct.

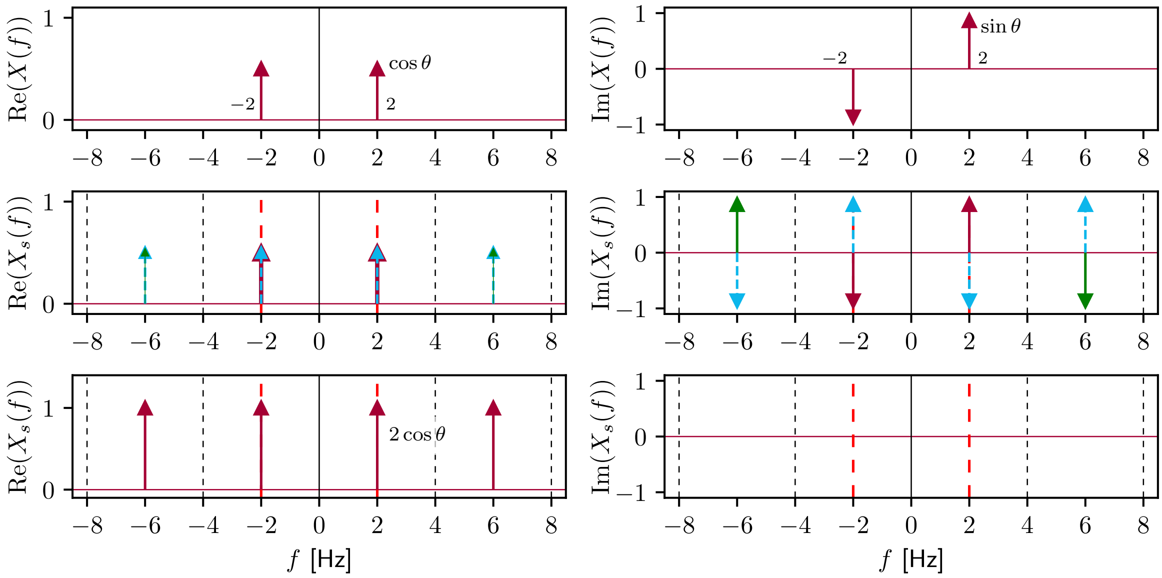


S.9-14 (a)-(c) The Fourier transform of $x(t)$ equals

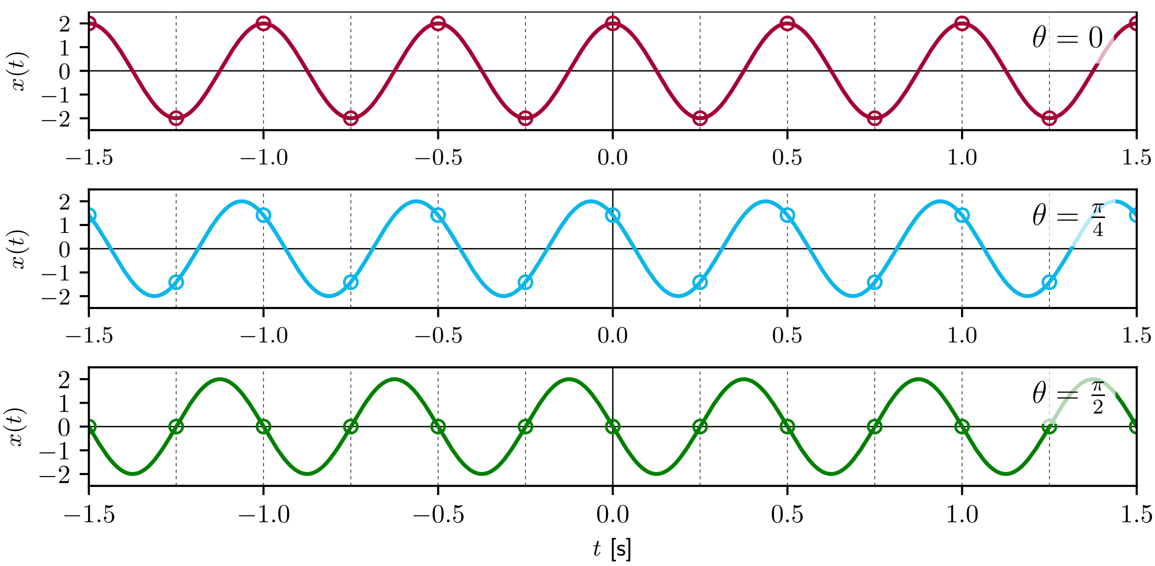
$$X(f) = \cos\theta\big(\delta(f+2) + \delta(f-2)\big) - j\sin\theta\big(\delta(f+2) - \delta(f-2)\big)$$

using (A.7), (5.16) and (5.17). The highest frequency occurring in $X(f)$ is 2 Hz, the Nyquist rate of this signal is 4 Hz. Because $X(f = f_h = 2) \neq 0$, to prevent aliasing, this signal must be sampled with a sampling frequency that is strictly larger than its Nyquist rate, see (9.12).

In this exercise we are instructed to sample at the Nyquist rate, i.e., $f_s = 4$ Hz, and study what happens. In the following figure (for $\theta = \frac{\pi}{3}$) the Fourier transform $X(f)$ is shown at top, the construction of $X_s(f)$ in the middle (with the copies of $X(f)$ at integer multiples of $f_s = 4$ Hz in different colors), and the final result, the Fourier transform of the samples $X_s(f)$ at bottom. It is clear that, no matter the value of θ, the imaginary part of $X_s(f)$ will be zero: the positive and negative Dirac pulses of the copies will cancel each other. The real part of $X_s(f)$ consists of a Dirac comb: an infinite array of Dirac pulses, all with weight $2\cos\theta$. When $\theta = \frac{\pi}{2}$, also the real part of $X_s(f)$ is zero: all samples are zero.



In time, one can see what happens when sampling the sinusoidal signal $x(t)$ at its Nyquist rate. The figure below shows $x(t)$ and its samples (the circles) for the three values of θ; the sampling times at integer multiples of $\Delta t = \frac{1}{f_s} = \frac{1}{4}$ s are shown using the black dashed vertical lines. For $\theta = 0$, the cosine is sampled at its minimum and maximum values, the samples alternate between $+2$ and -2. For $\theta = \frac{\pi}{4}$, the samples alternate between $\pm\sqrt{2}$. For $\theta = \frac{\pi}{2}$, the cosine is sampled at its zero crossings: all samples are zero.



This last result also follows from substituting $t = n\Delta t$ with $n \in \mathbb{Z}$ into the expression for $x(t)$:

$$x(t) = 2\cos(4\pi t + \theta) = 2\cos\theta\cos(4\pi t) - 2\sin\theta\sin(4\pi t)$$

$$\Rightarrow x(n\Delta t) = 2\cos\theta\cos\left(4\pi n\tfrac{1}{4}\right) - 2\sin\theta\sin\left(4\pi n\tfrac{1}{4}\right) = 2\cos\theta(-1)^n \;\; = 2(-1)^n\cos\theta$$

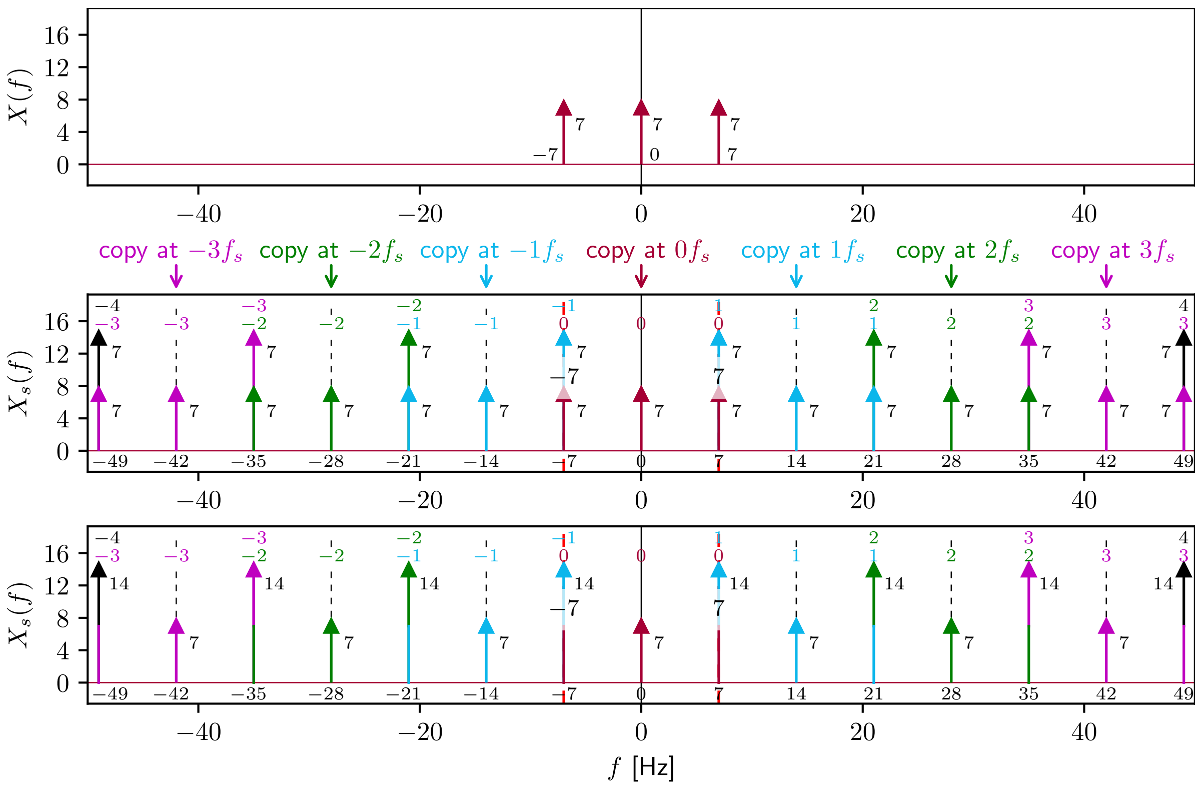
because $\sin(n\pi) = 0\,\forall\, n \in \mathbb{Z}$. After sampling at the Nyquist rate, regardless of the value of θ, the sine component has disappeared altogether, the sampled sequence will therefore always be even, its Fourier transform even and real-valued.

Sampling a sinusoidal signal exactly at its Nyquist rate means that it cannot be reconstructed from its samples (except when $\theta = 0$), as further discussed in Chapter 10. Note that we deliberately included this limit case for the sake of completeness, it has little practical relevance.

s.9-15 From Solutions *s*.9-8 and *s*.9-9 we know that for signal $x(t) = 7 + 14\cos(14\pi t)$ the highest frequency occurring in $X(f)$, f_h, equals 7 Hz, and that at this frequency, we have a Dirac pulse, i.e., $X(f = f_h) \neq 0$. Hence, for aliasing not to occur, we must apply a sampling frequency that is strictly larger than $2f_h$, i.e., $f_s > 14$ Hz. However, as in this case sampling frequency f_s equals 14 Hz, i.e., $f_s = 2f_h$, aliasing *does* occur. We sample $x(t)$ at the Nyquist rate and obtain a similar situation as in Exercise *e*.9-14.

The figure below shows the Fourier transform of the original signal $X(f)$ at the top. The second row, for the sake of explanation, shows how the Fourier transform of the sampled signal $X_s(f)$ is constructed. The third row then shows $X_s(f)$. Note that the different colors used for a single Dirac pulse are merely to indicate which copies 'contribute' to the pulse.

The table after the figure shows the frequencies at which Dirac pulses occur in $X_s(f)$. It can be seen that the 'left' Dirac pulse of copy $k = n$ and the 'right' Dirac pulse of copy $k = n-1$ end up at the same frequency. As a result, the Dirac pulses at kf_s Hz have a weight equal to the weight of the Dirac pulses in the Fourier transform of the original signal, $X(f)$. The other Dirac pulses have a weight that is twice the value of the weight of the original pulses.



k	kf_s	kf_s [Hz]	$f_{\text{Dirac\_1}}$ [Hz]	$f_{\text{Dirac\_2}}$ [Hz]	$f_{\text{Dirac\_3}}$ [Hz]
0	$0f_s$	0	$0-7 = -7$	$0+0 = 0$	$0+7 = 7$
1	$1f_s$	14	$14-7 = 7$	$14+0 = 14$	$14+7 = 21$
2	$2f_s$	28	$28-7 = 21$	$28+0 = 28$	$28+7 = 35$
3	$3f_s$	42	$42-7 = 35$	$42+0 = 42$	$42+7 = 49$
4	$4f_s$	56	$56-7 = 49$	$56+0 = 56$	$56+7 = 63$
-1	$-1f_s$	-14	$-14-7 = -21$	$-14+0 = -14$	$-14+7 = -7$
-2	$-2f_s$	-28	$-28-7 = -35$	$-28+0 = -28$	$-28+7 = -21$
-3	$-3f_s$	-42	$-42-7 = -49$	$-42+0 = -42$	$-42+7 = -35$
-4	$-4f_s$	-56	$-56-7 = -63$	$-56+0 = -56$	$-56+7 = -49$

When trying to reconstruct original signal $x(t)$ from its samples, and thereby using the frequency band from $-\frac{f_s}{2}$ to $\frac{f_s}{2}$ Hz, it would depend on the exact definition of the frequency band (i.e., whether the boundary frequencies $-\frac{f_s}{2}$ and $\frac{f_s}{2}$ Hz are included or excluded), how many Dirac pulses the Fourier transform of the reconstructed signal, $X_r(f)$, contains.

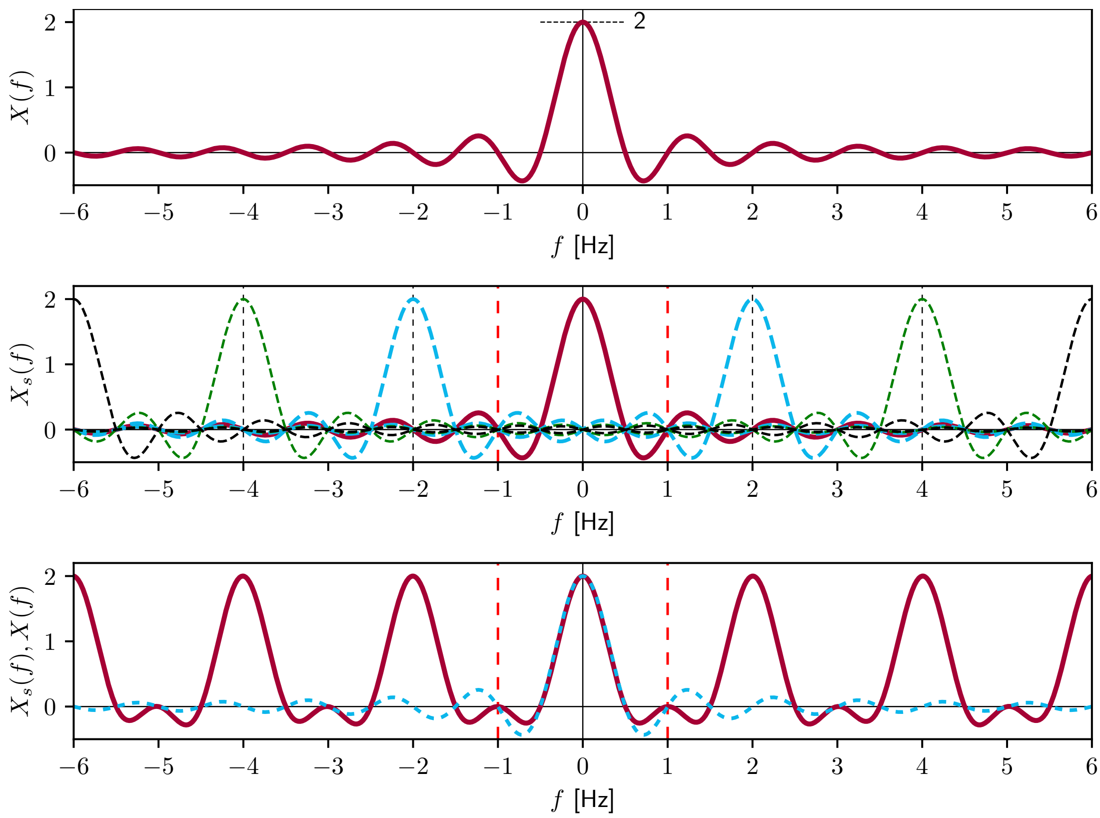
When using an ideal signal reconstruction filter, the subject of Chapter 10, it would theoretically be possible to reconstruct this signal from its samples. Such an ideal filter passes $X_s(f)$ from $-\frac{f_s}{2}$ and $\frac{f_s}{2}$ Hz, multiplies $X_s(f)$ by 0 for all other frequencies, and has a value of 0.5 exactly at its filter boundary frequencies $-\frac{f_s}{2}$ and $\frac{f_s}{2}$ Hz, (10.2), based on (B.1). This would mean that the weight of the Dirac pulses of $X_s(f)$ at these frequencies, $-\frac{f_s}{2}$ and $\frac{f_s}{2}$ Hz, i.e., 14, is multiplied by 0.5, restoring the original weight of 7 of $X(f)$.

This situation can be compared with the $\theta = 0$ case of Exercise e.9-14, where the cosine signal with $\theta = 0$ could be reconstructed from its samples even when sampling this signal at its Nyquist rate (see Solution s.9-14). However, would the cosine component in $x(t)$ in the current exercise have just a tiny bit of phase, we would *not* be able to reconstruct it from its samples when sampling at its Nyquist rate. Again, we have discovered an interesting limit case.

s.9-16

(a) The highest frequency occurring in signal $x(t)$ cannot be determined because its Fourier transform $X(f)$, when disregarding the zero-crossings at $k\frac{1}{\tau}$ Hz with $k \in \mathbb{Z}\setminus\{0\}$, will only become 0 when $|f| \to \infty$. This is the essence of a non-band-limited signal.

(b) The figure below shows $X(f)$ (at top), $X_s(f)$ (middle, with different copies of $X(f)$ in different colors) and $X_s(f)$ (adding all the copies, at bottom), for $A = 1$, $\tau = 2$ and $f_s = 2$ Hz. Of course, we get aliasing, as $X(f)$ is non-zero beyond the Nyquist frequency $\frac{f_s}{2} = 1$ Hz. $X(f)$ is much broader than the assigned frequency band with width f_s, see page 100 of the companion book, Theory. Within the range $[-\frac{f_s}{2}, \frac{f_s}{2})$, the central lobe of $X_s(f)$ coincides with that of $X(f)$; outside this lobe, but still within $[-\frac{f_s}{2}, \frac{f_s}{2})$, $X_s(f)$ differs from $X(f)$ (blue dashed curve at bottom).

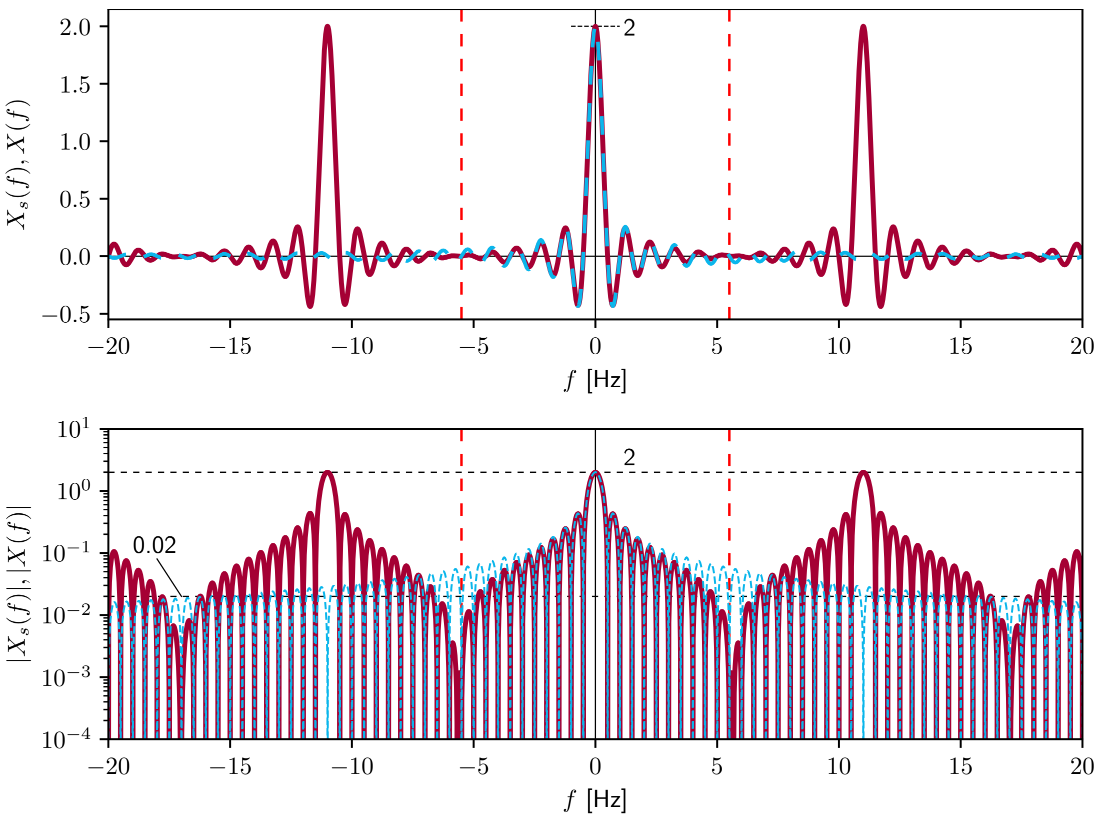


(c) We could argue that when the magnitude of $X(f)$, $|X(f)|$, is 'small enough' (again, disregarding the zero-crossings in $X(f)$), that then the inevitable effects of aliasing would be acceptably small. This would require us to calculate $|X(f)|$ and see where the local maximum in this function (between two zero-crossings at $\frac{k}{\tau}$ Hz and $\frac{k+1}{\tau}$ Hz) becomes smaller than some threshold value ϵ. Then we could define the highest frequency occurring in the signal, f_h, to be $\approx \frac{k}{\tau}$ Hz and could sample at $2f_h$.

We could also consider this problem from the perspective of the signal's energy. That is, we could argue that when the contribution of a frequency band $[-f_1, f_1]$ for some frequency f_1 to the total signal energy would be, for instance, 99%, then $f_h \approx f_1$ and we could sample at $2f_h$.

In any case, some aliasing will always occur, as $X(f)$ will be non-zero beyond the Nyquist frequency $\frac{f_s}{2}$. Taking a (very) high f_s mitigates the problem of aliasing, but may lead to an excessive number of samples, draining computer memory and storage.

(d) In Solution S.5-12(d) we derived that for an arbitrary pulse with amplitude A and width τ s, more than 99% of the signal's total energy lies in the frequency band from $-\frac{11}{\tau}$ to $\frac{11}{\tau}$ Hz. Hence, in the current context, $f_1 = \frac{11}{2} = 5.5$ Hz, $f_h \approx f_1$ and the required sampling frequency would be $f_s = 2f_h = 11$ Hz. The figure below shows $X_s(f)$ for this sampling frequency (in burgundy), together with $X(f)$ (dashed blue), using a linear (top) and logarithmic (bottom) vertical axis. Note that when using a logarithmic axis, only positive values can be plotted, therefore $|X_s(f)|$ and $|X(f)|$ are shown.



Clearly, in the figure above $X_s(f)$ more closely resembles $X(f)$ within the range $[-\frac{f_s}{2}, \frac{f_s}{2})$ than with a sampling frequency $f_s = 2$ Hz, but some aliasing still occurs. On the logarithmic vertical axis, two black dashed horizontal lines are plotted. One line represents the peak value 2 of $X(f)$, the other line represents $\frac{1}{100}$th of that peak value. We can see that near the limits of the range, $f = \pm\frac{f_s}{2}$, $|X_s(f)|$ is much smaller than $|X(f)|$.

10

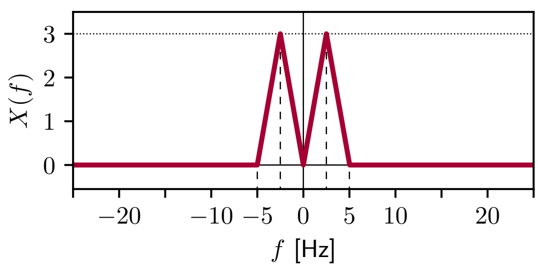
Signal reconstruction

Exercises

e|10.1 Ideal signal reconstruction

e.10-1

Signal $x(t)$ with Fourier transform $X(f)$ as shown on the right, is sampled with sampling frequency f_s, and then reconstructed again using an ideal reconstruction filter $H_r(f)$, perfectly passing signal components at all frequencies between $-\frac{f_s}{2}$ and $+\frac{f_s}{2}$ Hz and perfectly blocking the signal components at all other frequencies.



For each of the three sampling frequencies specified below:

(1) Determine whether aliasing occurs.

(2) Draw Fourier transform $X_s(f)$ of sampled signal $x_s(t)$.

(3) Draw Fourier transform $X_r(f)$ of reconstructed signal $x_r(t)$.

(a) $f_s = 15$ Hz (b) $f_s = 10$ Hz (c) $f_s = 7.5$ Hz

e.10-2

In this exercise we examine a 'limit case' in sampling. Consider signal

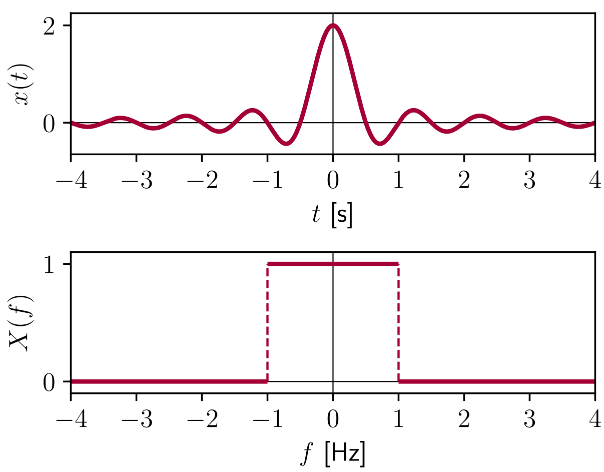
$$x(t) = 2\,\text{sinc}(2t)$$

with Fourier transform

$$X(f) = \Pi\left(\frac{f}{2}\right), \qquad (6.13)$$

both shown on the right. The signal's Nyquist rate is $2f_h = 2$ Hz.

Investigate what happens when you sample $x(t)$ exactly with sampling frequency $f_s = 2$ Hz, then reconstruct the signal using an ideal reconstruction filter, as in (10.2).

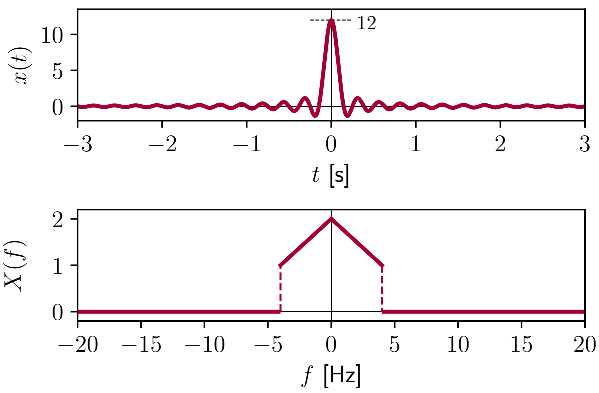


e.10-3 We investigate another 'limit case' in sampling. Consider signal

$$x(t) = 4\,\mathrm{sinc}^2(4t) + 8\epsilon\,\mathrm{sinc}(8t)$$

with Fourier transform $X(f)$, both shown on the right, for $\epsilon = 1$. The Nyquist rate of this signal is 8 Hz.

Investigate what happens when you sample $x(t)$ exactly with sampling frequency $f_s = 8$ Hz, then reconstruct the signal using an ideal reconstruction filter, as in (10.2).



e| **10.c** Computer exercises

e.10-4 In Exercise e.9-16 it was investigated what the effects are when sampling a non-band-limited signal, namely pulse $x(t) = A\Pi\left(\frac{t}{\tau}\right)$, with $A = 1$ and $\tau = 2$. In this exercise, we investigate what happens when reconstructing this signal after sampling. Suppose the signal was sampled at the sampling frequencies given below. For each case, reconstruct the continuous-time pulse using an ideal reconstruction filter and explain the result.

(a) $f_s = 2$ Hz (b) $f_s = 11$ Hz (c) $f_s = 101$ Hz

e.10-5 Consider signal

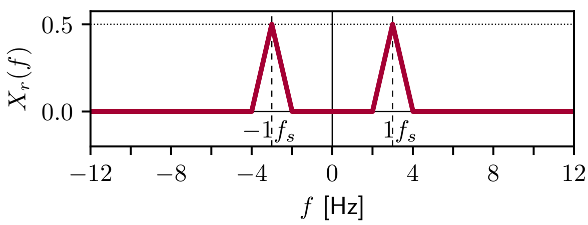
$$x(t) = \mathrm{sinc}^2(t)$$

with Fourier transform

$$X(f) = \Lambda(f) \qquad \text{cf. (7.7), with } \tau = 1,$$

as shown in Figure 9.3. The highest frequency present in this signal, f_h, is 1 Hz. In Figure 9.5, signal $x(t)$ was sampled at $f_s = 3$ Hz, and the resulting Fourier transform $X_s(f)$ was shown in Figure 9.6: copies of the original Fourier transform $X(f)$ occur at every integer multiple k of the sampling frequency f_s, with $k \in \mathbb{Z}$. In this case, the Nyquist sampling theorem, (9.12), is satisfied: the copies do not overlap, and aliasing does not occur. The signal can be perfectly reconstructed using the *central* part of $X_s(f)$ from $f = -\frac{f_s}{2}$ to $\frac{f_s}{2}$, as indicated by the red dashed vertical lines in Figure 9.6.

In this exercise, we wonder what happens if we were to reconstruct the signal (instead) from one of the copies in the Fourier transform $X_s(f)$ of the sampled signal, e.g., from $X(f - kf_s)$ for $k = 1$. Starting from a real-valued Fourier transform $X_s(f)$ in this case, we choose to maintain a symmetric Fourier transform $X_r(f)$, as to obtain a real-valued reconstructed signal $x_r(t)$, with amplitude identical to the original signal $x(t)$. Therefore, consider the copy of $X(f)$ at $1f_s$ and the copy at $-1f_s$, and the energy equally divided between the two, as in the figure on the right.



(a) Reconstruct signal $x_r(t)$ from Fourier transform $X_r(f)$.

(b) Plot the reconstructed signal $x_r(t)$.

(c) Compare the reconstructed signal $x_r(t)$ with the original signal $x(t) = \mathrm{sinc}^2(t)$.

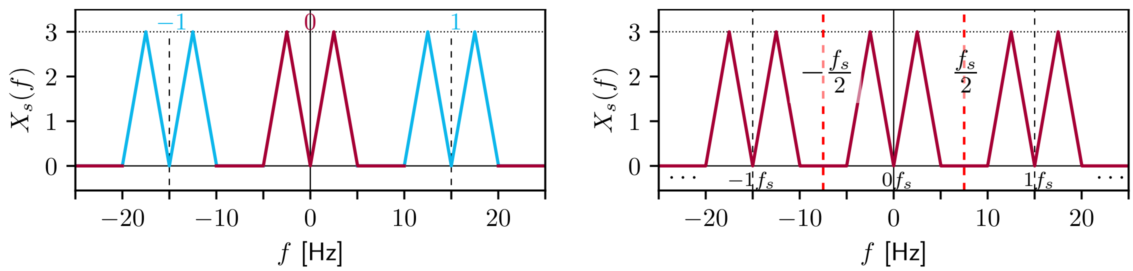
Solutions

s|10.1 Ideal signal reconstruction

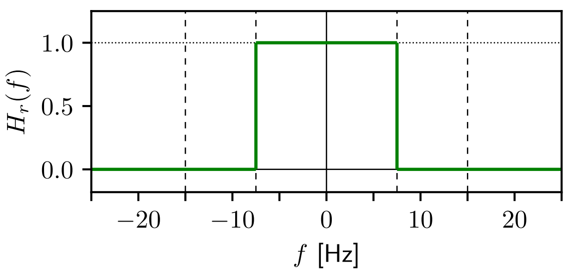
s.10-1

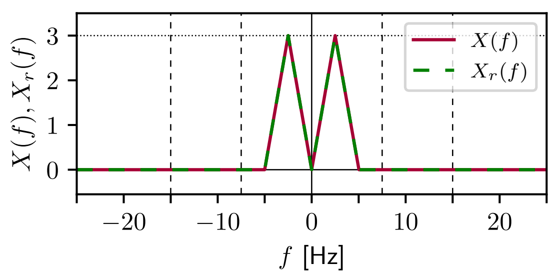
The illustration in Exercise *e*.10-1 shows that the highest frequency of $X(f)$, f_h, equals 5 Hz. Hence, the signal's Nyquist rate equals $2f_h = 2 \cdot 5 = 10$ Hz. At $f_h = 5$ Hz, $X(f = f_h) = 0$, meaning that sampling with a sampling frequency equal to or higher than $2f_h = 10$ Hz yields a sampled signal $x_s(t)$ that can later be used to perfectly reconstruct original signal $x(t)$, (9.14). Sampling with a rate lower than 10 Hz will lead to aliasing, and then original signal $x(t)$ cannot be reconstructed anymore from its samples.

(a) Sampling frequency $f_s = 15$ Hz is higher than $2f_h = 10$ Hz; aliasing does *not* occur.

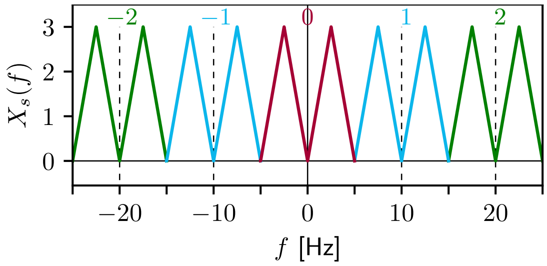


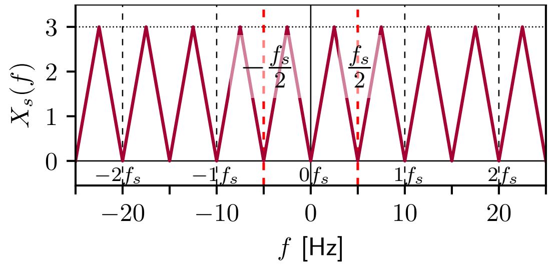
Fourier transform $X_s(f)$ of sampled signal $x_s(t)$ is shown for $f_s = 15$ Hz. On the left, the different copies (repetitions of Fourier transform $X(f)$ at integer multiples of f_s) are shown in different colors. At the top right, the final spectrum $X_s(f)$ is shown. Original signal $x(t)$ can be perfectly reconstructed using an ideal (low-pass, see Section 18.1) reconstruction filter $H_r(f)$ with one-sided filter bandwidth $\frac{f_s}{2} = 7.5$ Hz, shown at the middle right. Fourier transform $X_r(f)$ of reconstructed signal $x_r(t)$ is shown at the bottom right and is identical to original Fourier transform $X(f)$.



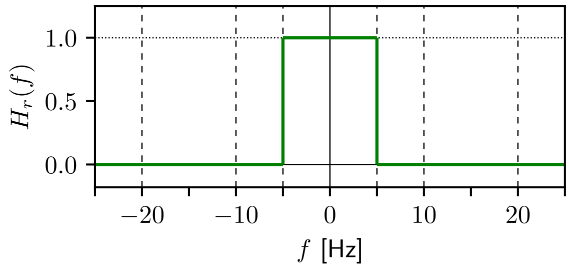


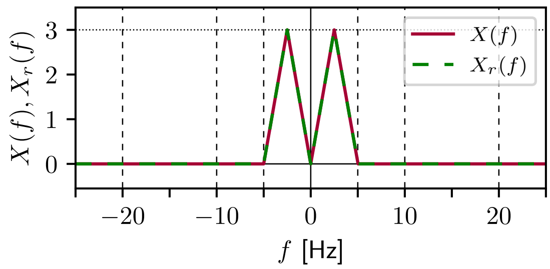
(b) Sampling frequency $f_s = 10$ Hz equals the signal's Nyquist rate of $2f_h = 10$ Hz, hence, aliasing does *not* occur. Original signal $x(t)$ can be perfectly reconstructed again using an ideal reconstruction filter $H_r(f)$ with one-sided filter bandwidth $\frac{f_s}{2} = 5$ Hz.



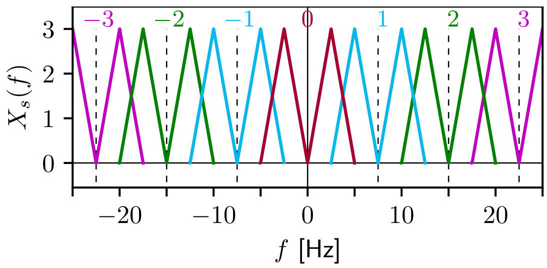


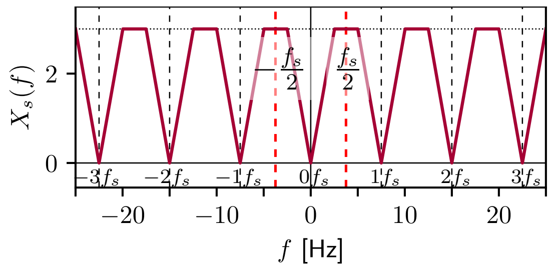
However, as one can see from $X_s(f)$, here we are 'just safe'. The original signal can still be perfectly reconstructed, but only using an ideal (low-pass) reconstruction filter $H_r(f)$ with a filter bandwidth of exactly $\frac{f_s}{2} = 5$ Hz, as is the case here. Any other reconstruction filter cannot be used *without* either losing some of the information in $x(t)$ (i.e., filtering away part of the signal in the frequency range $|f| < 5$ Hz, see Chapter 18), or distorting the signal by copying part of the (first) alias energy and using that to reconstruct the signal. Therefore, it is not recommended to sample *at* the Nyquist rate.



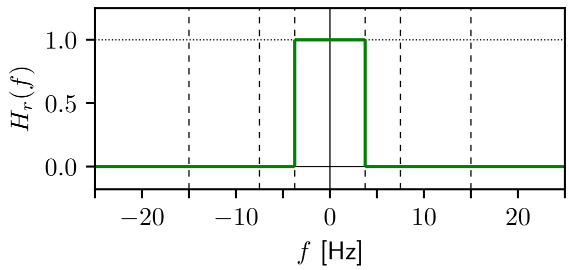


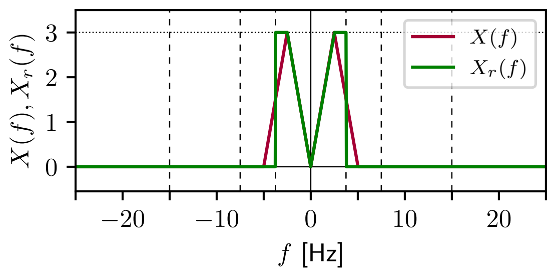
(c) Clearly, sampling frequency $f_s = 7.5$ Hz is lower than $2f_h = 10$ Hz; aliasing *does* occur.





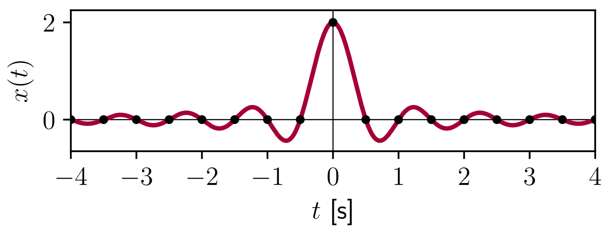
As shown at the top left, the copies of Fourier transform $X(f)$ overlap. The energy in the copies merges, resulting in Fourier transform of the sampled signal $X_s(f)$ shown at the rop right. Shown at the bottom right in green is Fourier transform $X_r(f)$ of reconstructed signal $x_r(t)$, which differs from the original Fourier transform $X(f)$ in burgundy. Hence, original signal $x(t)$ cannot be reconstructed; $x_r(t)$ will *not* equal $x(t)$. No filter exists that can reconstruct the original signal from its samples. There will always be energy from the alias in the spectrum, which we cannot distinguish from the original signal's energy.



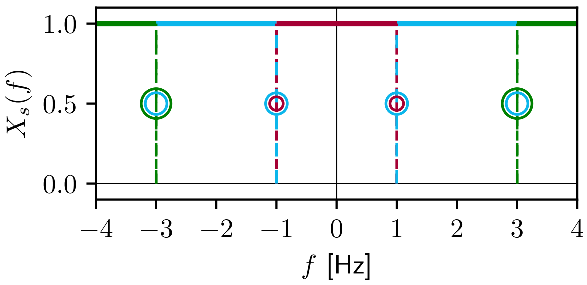


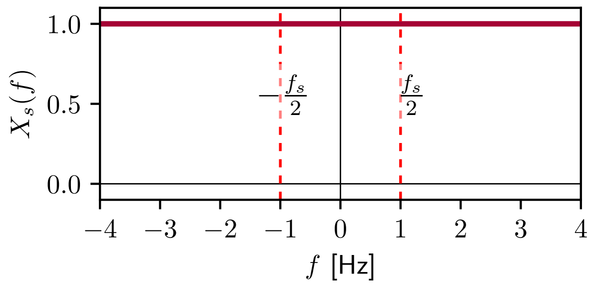
s.10-2

With $f_s = 2$ Hz, resulting in samples at integer multiples of $\Delta t = \frac{1}{2}$ s, we sample exactly at the Nyquist rate. The figure on the right shows that these samples (black dots) coincide *exactly* with all the zero-crossings of $x(t)$, except for the sample at $t = 0$ s.

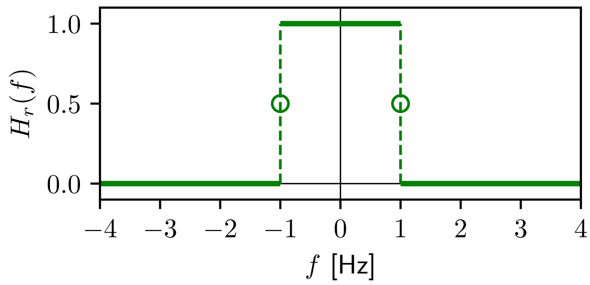


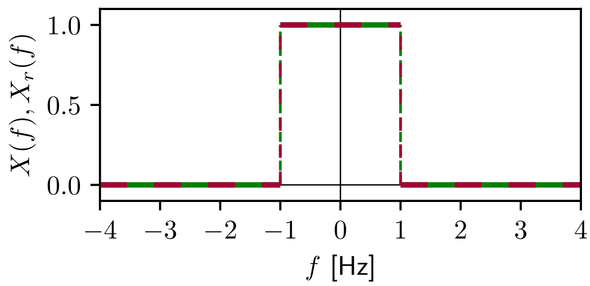
Since here we deliberately sample *at* the Nyquist rate, we end up with a limit case.





Fourier transform $X_s(f)$ of sampled signal $x_s(t)$ is shown for $f_s = 2$ Hz. On the left, the different copies (repetitions of Fourier transform $X(f)$ at integer multiples of f_s) are shown in different colors. The burgundy, blue and green circles indicate the values of the copies of $X(f)$ at the edges of the pulse in the frequency domain, all equal to 0.5, (B.1); these are explicitly shown here to accentuate the fact that we are investigating a limit case. The highest frequency of $X(f)$, f_h, equals 1 Hz, and *at* $f = f_h$, $X(f) = 0.5 \neq 0$. At the top right, the final spectrum $X_s(f)$ is shown; when adding the values of the copies of $X(f)$ at the edges, we obtain $0.5 + 0.5 = 1$. The result is $X_s(f) = 1\,\forall f$. Original signal $x(t)$ can still be perfectly reconstructed, however, using an ideal reconstruction filter $H_r(f) = \Pi\left(\frac{f}{2}\right)$, (10.2), with one-sided filter bandwidth $\frac{f_s}{2} = 1$ Hz, shown at the middle right, with the values at the edges of $H_r(f)$ also indicated with circles. Fourier transform $X_r(f)$ of reconstructed signal $x_r(t)$ is shown at the bottom right and is indeed identical to original Fourier transform $X(f)$. In frequency, $X_r(f) = H_r(f)X_s(f) = X(f)$, with a value of 0.5 at the frequencies $f = \pm 1$ Hz. In time, only at $t = 0$ s we have a non-zero sample; with $\Delta t = \frac{1}{2}$ s we obtain $x_s(t) = \delta(t)$, (9.2) and (9.3); the Dirac pulse has a weight equal to $\Delta t 2\,\mathrm{sinc}(0) = \frac{1}{2}2 = 1$. Then, $x_r(t) = h_r(t) * x_s(t) = x(t)$: the convolution of the sinc function $h_r(t) = \frac{1}{\Delta t}\,\mathrm{sinc}\left(\frac{t}{\Delta t}\right) = 2\,\mathrm{sinc}(2t)$, (10.4), with $x_s(t) = \delta(t)$ yields $x_r(t) = 2\,\mathrm{sinc}(2t) = x(t)$.





We have a situation where $X(f) \neq 0$ for $f = f_h$ and, to be able to perfectly reconstruct the signal from its samples, we would expect the minimum sampling frequency f_s to be strictly larger than $2f_h$, see page 102 of the companion book, Theory. But apparently, in this case, we can still perfectly reconstruct the signal from its samples using the ideal reconstruction filter, when sampling at $f_s = 2f_h$. Earlier we have seen, e.g., in Exercises *e*.9-14 and *e*.9-15, that when there is a Dirac pulse at $f = f_h$ we can *not* reconstruct the signal when sampling at the Nyquist rate (see Solutions $\mathcal{S}$.9-14 and $\mathcal{S}$.9-15).

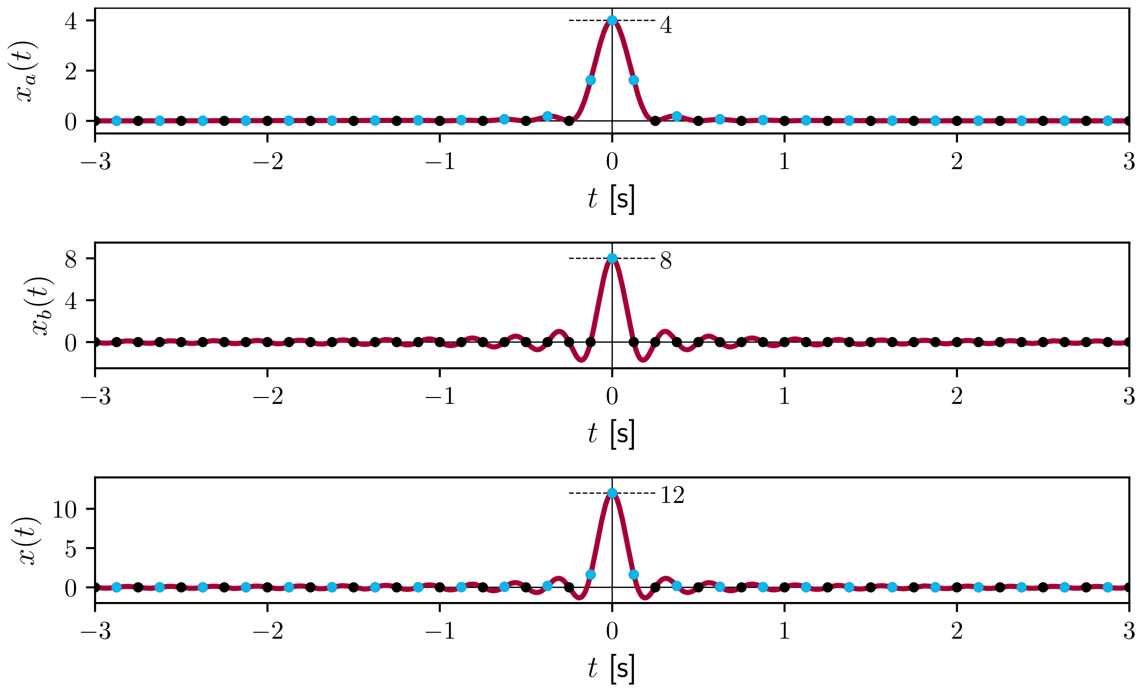
From this exercise we conclude that *as long as it is not a Dirac pulse* occurring at $f = f_h$, ideal reconstruction *does* allow the Fourier transform of a signal to have a non-zero value at $f = f_h$, and we *can sample at the Nyquist rate*. We study this phenomenon once more in the next exercise.

$\mathcal{S}$.10-3 Signal $x(t)$ can be written as the sum of two components: $x(t) = \underbrace{4\,\mathrm{sinc}^2(4t)}_{x_a(t)} + \underbrace{8\epsilon\,\mathrm{sinc}(8t)}_{x_b(t)}$.

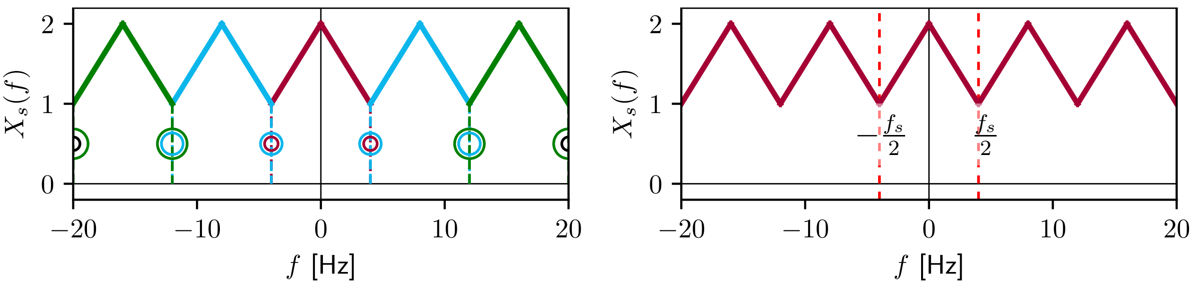
Then $X(f) = X_a(f) + X_b(f)$, with $X_a(f) = \Lambda\left(\frac{f}{4}\right)$ and $X_b(f) = \epsilon\Pi\left(\frac{f}{8}\right)$, using, respectively, (7.7) and (6.13). The highest frequency in $X_a(f)$ is 4 Hz, where $X_a(f = 4) = 0$. The Nyquist rate of this first component is 8 Hz, and because its Fourier transform equals 0 at $f = 4$ Hz it can be safely sampled with 8 Hz, cf. (9.14).

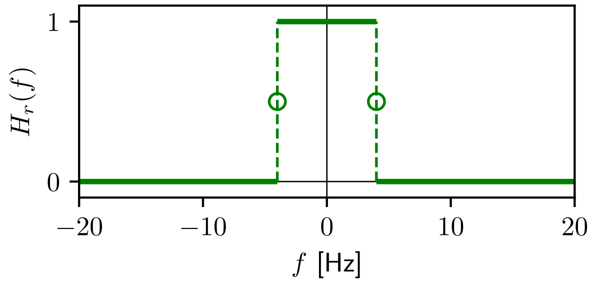
Component $x_b(t)$, with Fourier transform $X_b(f)$, is similar to the signal defined in Exercise $\mathcal{E}$.10-2; the highest frequency in $X_b(f)$ is also 4 Hz, but *at* this frequency, $X_b(f = 4) = \frac{\epsilon}{2} \neq 0$ for $\epsilon \neq 0$, (B.1). While the Nyquist rate of this component is also 8 Hz, according to the sampling theorem we should be sampling this component with a sampling frequency f_s strictly larger than 8 Hz, cf. (9.12). Exercise $\mathcal{E}$.10-2 has shown, however, that when ideal signal reconstruction is assumed, this component *can* be safely sampled at exactly 8 Hz (see Solution $\mathcal{S}$.10-2).

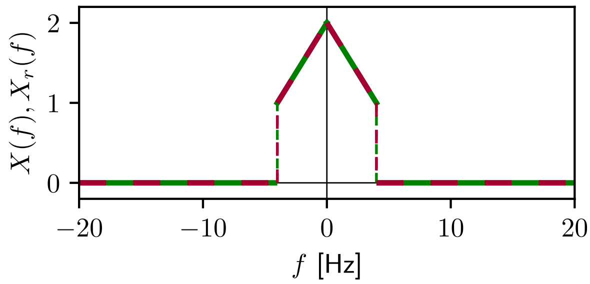
The figure below shows $x(t)$ (bottom) and its two components, $x_a(t)$ (top) and $x_b(t)$ (middle), for $\epsilon = 1$. Signal component $x_a(t)$ has its zero-crossings at integer multiples of $\frac{1}{4}$ s (except at $t = 0$ s). Therefore, when sampling at 8 Hz, $f_s = 2f_h$, the samples obtained correspond to the zero-crossings (black dots) and the intermediate values (blue dots). Signal component $x_b(t)$ has its zero-crossings at integer multiples of $\frac{1}{8}$ s (except at $t = 0$ s). Therefore, when sampling at 8 Hz, the samples obtained correspond exactly to the zero-crossings (black dots), and just one non-zero sample is obtained (blue dot). Together, we obtain sufficient non-zero samples of $x(t)$ to reconstruct the signal using an ideal filter.



Deliberately sampling *at* the Nyquist rate, we end up with another limit case.





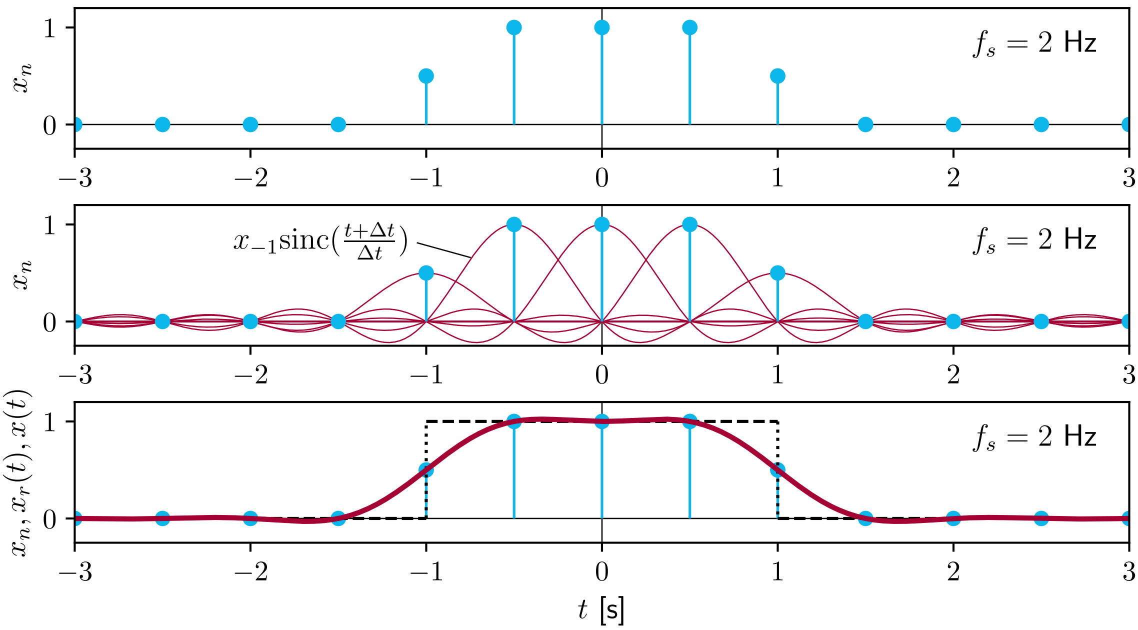


Fourier transform $X_s(f)$ of sampled signal $x_s(t)$ is shown for $f_s = 8$ Hz. On the left, the different copies (repetitions of Fourier transform $X(f)$ at integer multiples of f_s) are shown in different colors. The burgundy, blue, green and black circles indicate the values of the copies of $X(f)$ at the edges of the pulse in the frequency domain, all equal to $\frac{\epsilon}{2}$, (B.1). At the top right, the final spectrum $X_s(f)$ is shown; when adding the values of the copies of $X(f)$ at the edges, we obtain $\frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. Original signal $x(t)$ can be perfectly reconstructed using an ideal reconstruction filter $H_r(f) = \Pi\left(\frac{f}{8}\right)$, (10.2), with one-sided filter bandwidth $\frac{f_s}{2} = 4$ Hz, shown at the middle right, with the values at the edges of $H_r(f)$ also indicated with circles. Fourier transform $X_r(f)$ of reconstructed signal $x_r(t)$ is shown at the bottom right and is indeed identical to original Fourier transform $X(f)$. In frequency, $X_r(f) = H_r(f)X_s(f) = X(f)$, with a value of $\frac{\epsilon}{2}$ at the frequencies $f = \pm 4$ Hz.

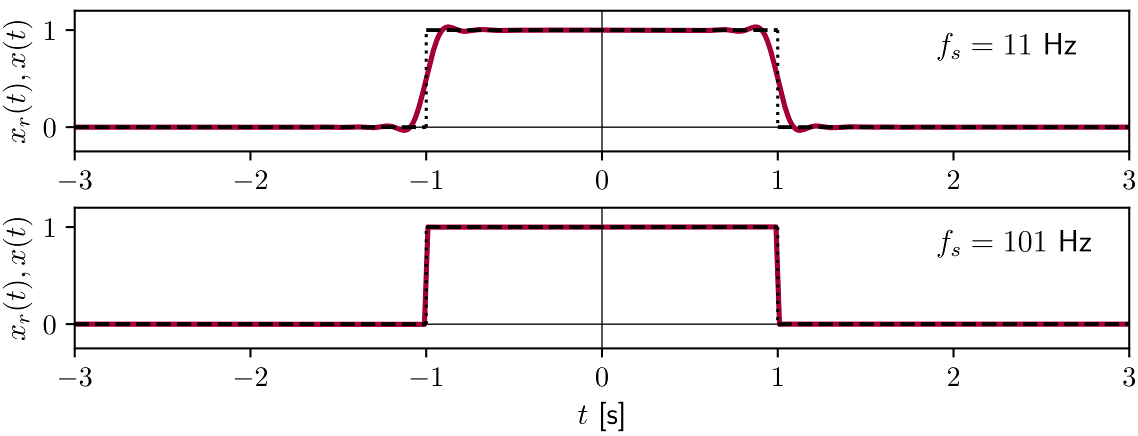
Apparently, for a band-limited signal and assuming ideal signal reconstruction, we *can* safely sample at the Nyquist rate even when $X(f)$ is non-zero at $f = f_h$, as long as it is not a Dirac pulse at this frequency.

S | **10.c** Computer exercises

(a) The figure below shows how the pulse is reconstructed using an ideal reconstruction filter, for $f_s = 2$ Hz. The top graph shows x_n, the middle graph shows the multiplication of each sample by Δt times the shifted sinc function $h_r(t) = f_s \operatorname{sinc}(f_s t) = 2\operatorname{sinc}(2t)$, (10.4), which is the inverse Fourier transform of the ideal reconstruction filter $H_r(f) = \Pi\left(\frac{f}{f_s}\right) = \Pi\left(\frac{f}{2}\right)$, (10.2). The bottom graph shows the result of adding all these sinc functions together, (10.6), yielding $x_r(t)$ shown in burgundy. For the sake of comparison, the original pulse $x(t)$ is shown using black dashed lines. We see that for each sample x_n, at $t = n\Delta t$ with $n \in \mathbb{Z}$, $x_r(t)$ exactly equals $x(t)$. Note that the pulse is also sampled at its edges, where it equals 0.5, not 1, (B.1). *S*.10-4



(b)-(c) The figure below shows the reconstructed pulse $x_r(t)$ for $f_s = 11$ Hz (top) and $f_s = 101$ Hz (bottom), together with the original pulse $x(t)$ in black dashed lines.



It is clear that a higher sampling frequency f_s allows for a better reconstruction of the pulse. The reconstructed signal $x_r(t)$ will, however, never exactly replicate the original pulse $x(t)$.

Basic Python code:

```
import numpy as np
from matplotlib import pyplot as plt

burgundy = [165/255,0/255,52/255]     # burgundy color

# pulse signal x(t) with amplitude A and width tau s
A   = 1                               # pulse amplitude [-]
tau = 2                               # pulse width [s]

# continuous time approximated by Delta_t = 0.01 s
fsct = 100; dtct = 1/fsct; Tend = 3; Nct = fsct*Tend*2
t    = np.linspace(-Tend, Tend, Nct) # time vector from -3 to 3 s
```

```
# define samples for f_s = 2 Hz
fs   = 2                        # sampling frequency [Hz]
dt   = 1/fs                     # time interval or time step size [s]
Ndt  = fs*Tend*2                # number of samples
td2= np.linspace(-Tend, Tend, Ndt+1) # time vector of sampled signal
xdt2 = A*np.heaviside(td2+tau/2,1)-A*np.heaviside(td2-tau/2-dt,1)# samples
xdt2[fs*Tend-fs,] = A/2          # at edges of pulse value equals A/2
xdt2[fs*Tend+fs,] = A/2

# multiply all samples by
# (approximated continuous-time) sinc function
xr2 =np.zeros(t.shape)          # signal with only zero samples
for kk in range(Ndt+1):
    # put the sinc at the right position in time
    hlp = np.sinc((t-(kk-Ndt/2)*dt)/dt)
    # add sinc to the reconstructed signal
    xr2 = xr2+hlp*xdt2[kk,]

# plot how signal x(t) is reconstructed
plt.figure()
# plot samples
plt.subplot(3,1,1)
plt.stem(td2, xdt2, basefmt=' ', linefmt='b', markerfmt='b')
plt.xlabel('$t$ [s]'); plt.ylabel('$x_n$')
# plot multiplication of samples by (approximated continuous-time) shifted
    sinc function
plt.subplot(3,1,2)
for kk in range(Ndt+1):
    # put the sinc at the right position in time
    hlp = np.sinc((t-(kk-Ndt/2)*dt)/dt);
    plt.plot(t, hlp*xdt2[kk,], color=burgundy, linewidth=0.5)
plt.stem(td2, xdt2, basefmt=' ', linefmt='b', markerfmt='b')
plt.xlabel('$t$ [s]'); plt.ylabel('$x_n$')
# plot reconstructed signal x_r(t) together with original pulse x(t)
plt.subplot(3,1,3)
plt.stem(td2, xdt2, basefmt=' ', linefmt='b', markerfmt='b')
plt.plot([-3,-1],[0,0],'k',linestyle='--',linewidth=1.2)
plt.plot([-1,-1],[0,1],'k',linestyle=':', linewidth=1.2)
plt.plot([-1, 1],[1,1],'k',linestyle='--',linewidth=1.2)
plt.plot([ 1, 1],[1,0],'k',linestyle=':', linewidth=1.2)
plt.plot([ 1, 3],[0,0],'k',linestyle='--',linewidth=1.2)
plt.plot(t, xr2, color=burgundy, linewidth=2)
plt.xlabel('$t$ [s]'); plt.ylabel('$x_n, x_r(t), x(t)$')

# the remainder of the code for (b) and (c) is highly similar to the code
# above for (a) and is therefore omitted to avoid excessive repetition
```

s.10-5

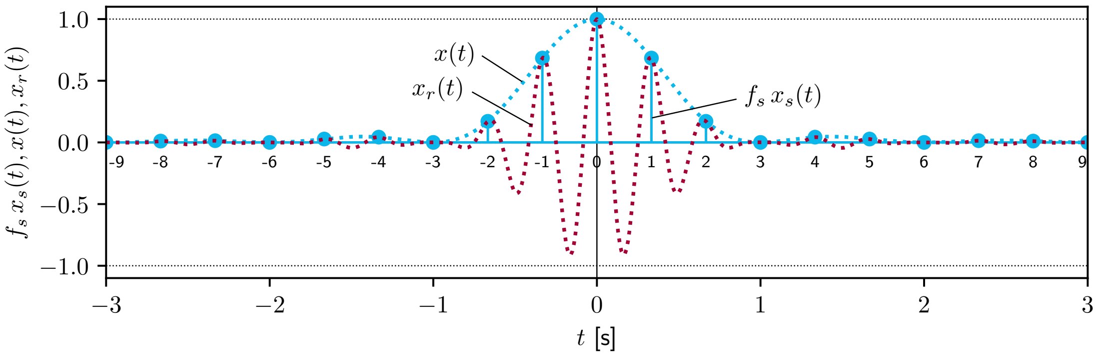
(a) The expression for $X_r(f)$, shown in Exercise e.10-5, reads:

$$X_r(f) = \tfrac{1}{2}\Lambda(f+f_s) + \tfrac{1}{2}\Lambda(f-f_s) \quad \text{with} \quad f_s = 3 \text{ Hz}$$

Using Fourier transform pair $\tau\,\text{sinc}^2(\tau t) \overset{\mathcal{F}}{\leftrightarrow} \Lambda\left(\frac{f}{\tau}\right)$, (7.7), the frequency translation theorem (6.6), and $\frac{1}{2}\left(e^{-j\theta}+e^{j\theta}\right) = \cos\theta$, (C.2), we obtain:

$$x_r(t) = \tfrac{1}{2}\,\text{sinc}^2(t)\,e^{-j2\pi f_s t} + \tfrac{1}{2}\,\text{sinc}^2(t)\,e^{j2\pi f_s t} = \text{sinc}^2(t)\cos(2\pi f_s t)$$

(b) The figure below shows (infinite-duration) signal $x(t)$ in dotted blue. The model of the samples, $x_s(t)$ multiplied by $f_s = 3$ Hz, is illustrated by the blue stems, as in Figure 9.5. The reconstructed signal $x_r(t)$ is shown in dotted burgundy.



(c) We do get the original signal $x(t) = \text{sinc}^2(t)$, though additionally multiplied by a cosine with frequency $f_s = 3$ Hz. This causes $x_r(t)$ to oscillate, see the figure in (b), though $x_r(t)$ (in burgundy) and $x(t)$ (in blue) do meet perfectly at the sample times $t = n\Delta t$ with $\Delta t = \frac{1}{3}$ s and $n \in \mathbb{Z}$, as then $\cos(2\pi f_s n\Delta t) = \cos(2\pi n) = 1$.

This exercise demonstrates the *ambiguity* that sampling brings about. Without prior information, and just relying on the samples of a signal, one could choose any of the copies $X(f - kf_s)$ in $X_s(f)$ to reconstruct the signal, $k \in \mathbb{Z}$, and one will find that the reconstructed (continuous-time) signal $x_r(t)$ is generally different from $x(t)$, though they do meet perfectly at the sample times $t = n\Delta t$. Based on just the samples, there are multiple options to reconstruct the signal, which is the key point of Figure 9.2.

Basic Python code:

```
import numpy as np
from matplotlib import pyplot as plt

burgundy = [165/255,0/255,52/255]          # burgundy color

# continuous-time signal x(t) and reconstructed signal xr(t)
t   = np.linspace(-3, 3, 601)           # time steps 0.01 s from -3 s to 3 s
fs  = 3                                  # sampling frequency [Hz]
xt  = np.sinc(t)**2                      # x(t) = sinc^2(t)
xrt = np.sinc(t)**2 * np.cos(2*np.pi*fs*t) # xr(t)=sinc^2(t)cos(2*pi*fs*t)

# discrete-time sampled signal fs*xs(t)
ts  = np.linspace(-3, 3, 19) # time steps 1/3 s from -3 s to 3 s (fs=3 Hz)
xts = np.sinc(ts)**2         # includes scaling with fs

# plot of x(t), fs*xs(t) and xr(t)
plt.figure()
plt.plot(t,  xt, ':', color='b', linewidth=1.5)
plt.stem(ts, xts, basefmt='b', linefmt='b', markerfmt='b')
plt.plot(t,  xrt, ':', color=burgundy, linewidth=1.5)
plt.xlabel('$t$ [s]'); plt.ylabel('$f_s\,x_s(t), x(t), x_r(t)$')
```

11

Discrete-Time Fourier Transform

Exercises

e| **11.1** Derivation of the DTFT

e.11-1

Prove that the DTFT of

$$x_n = \begin{cases} 1 & 0 \le n \le M-1 \\ 0 & \text{else} \end{cases}$$

equals

$$X_{\Delta t}(f) = \Delta t \, \frac{\sin\left(2\pi f \Delta t \left(\frac{M}{2}\right)\right)}{\sin\left(\frac{2\pi f \Delta t}{2}\right)} e^{-j2\pi f \Delta t\left(\frac{M-1}{2}\right)}$$

with $M \ge 1$ and $-\frac{f_s}{2} \le f < \frac{f_s}{2}$.

Note: this is the third DTFT pair in Table G.1.

e| **11.4** DTFT properties

e.11-2

Prove

$$X_{\Delta t}(-f) = X^*_{\Delta t}(f) \qquad (11.9)$$

for real-valued, infinite-duration sequences x_n.

Solutions

s| 11.1 Derivation of the DTFT

s.11-1 This DTFT can be proven through direct substitution of x_n in the expression for the DTFT (11.5):

$$X_{\Delta t}(f) = \Delta t \sum_{n=-\infty}^{\infty} x_n e^{-j2\pi f \Delta t n} = \Delta t \sum_{n=0}^{M-1} e^{-j2\pi f \Delta t n} = \Delta t \sum_{n=0}^{M-1} \left(e^{-j2\pi f \Delta t}\right)^n$$

On the right-hand side we obtain the geometric series identity, (A.22), with $a = e^{-j2\pi f \Delta t}$, $k = n$ and $N = M$. The geometric series identity states that we need to consider the cases $a = 1$ and $a \neq 1$. Here, for $f = 0$ we get $a = e^0 = 1$ so this case is considered first. For $f = 0$, $X_{\Delta t}(0) = \Delta t M$. For $f \neq 0$ we obtain:

$$X_{\Delta t}(f) = \Delta t \left(\frac{1 - e^{-j2\pi f \Delta t M}}{1 - e^{-j2\pi f \Delta t}}\right) \qquad (*)$$

The denominator can be written as $e^{-j\pi f \Delta t}\left(e^{j\pi f \Delta t} - e^{-j\pi f \Delta t}\right) = e^{-j\pi f \Delta t} 2j \sin\left(\frac{2\pi f \Delta t}{2}\right)$, (C.3). Similarly, the numerator can be written as $e^{-j\pi f \Delta t M} 2j \sin\left(\frac{2\pi f \Delta t M}{2}\right)$. Substitute in (*):

$$X_{\Delta t}(f) = \Delta t \, \frac{e^{-j\pi f \Delta t M}}{e^{-j\pi f \Delta t}} \, \frac{2j \sin\left(\frac{2\pi f \Delta t M}{2}\right)}{2j \sin\left(\frac{2\pi f \Delta t}{2}\right)} = \Delta t \, e^{-j2\pi f \Delta t \left(\frac{M-1}{2}\right)} \frac{\sin\left(2\pi f \Delta t \left(\frac{M}{2}\right)\right)}{\sin\left(\frac{2\pi f \Delta t}{2}\right)} \qquad \textbf{qed}$$

Because x_n is neither even, nor odd, $X_{\Delta t}(f)$ is a complex-valued function of f. For $f = 0$ this expression for $X_{\Delta t}(f)$ results in a $\frac{0}{0}$ case; using L'Hôpital's rule yields $X_{\Delta t}(0) = \Delta t M$, as found before. Note: for $M = 1$ we obtain the fourth DTFT pair in Table G.1.

s| 11.4 DTFT properties

s.11-2 The proof is very similar to the proof of (5.10) on page 59 of the companion book, Theory:

$$X^*_{\Delta t}(f) = \Delta t \sum_{n=-\infty}^{\infty} \left(x_n e^{-j2\pi f \Delta t n}\right)^* = \Delta t \sum_{n=-\infty}^{\infty} x^*_n e^{j2\pi f \Delta t n} = X_{\Delta t}(-f) \qquad \textbf{qed}$$

where we used (11.5) and the fact that $x^*_n = x_n$.

12

Discrete Fourier Transform

Note: in line with the companion book, Theory, as stated in its Section 12.2 on page 125, in this chapter and in subsequent chapters, the T index is omitted in the finite-length, discrete-time sequence: $x_{n,T}$ is written just as x_n.

Exercises

e| 12.2 Definition of the DFT and inverse DFT

e.12-1

Prove that the DFT of

$$x_n = \begin{cases} 1 & 0 \leq n \leq M-1 \\ 0 & \text{else} \end{cases}$$

equals

$$X_k = \Delta t \frac{\sin\left[k\frac{2\pi}{N}\left(\frac{M}{2}\right)\right]}{\sin\left[\frac{k\frac{2\pi}{N}}{2}\right]} e^{-jk\frac{2\pi}{N}\left(\frac{M-1}{2}\right)}$$

with $1 \leq M \leq N, \quad 0 \leq n \leq N-1 \quad \text{and} \quad 0 \leq k \leq N-1.$

Note: this is the first DFT pair in Table H.1.

e| 12.3 DFT frequencies, properties, and diagram

e.12-2

In this exercise, you will need to construct the DFT diagram of coefficients, illustrated in Figure 12.4, for three cases. For each case:

- Create the diagram showing the coefficients X_k as a function of index k.
- Indicate using which DFT coefficient the signal average can be computed.
- Show the positive frequencies and negative frequencies.
- If applicable, indicate which frequency index k corresponds to the Nyquist frequency.
- Show the relationship between X_4 and one of the other DFT coefficients (for you to determine which one), in a similar way as in Figure 12.4.
- Show the continuous frequency interval $[0, f_s)$, as shown in gray in Figure 12.4.

Note that knowing the sampling frequency f_s and measurement time T is irrelevant when constructing a DFT diagram. The three cases are:

(a) $N = 10$ (b) $N = 17$ (c) $N = 24$

e| 12.5 Two more examples: DFT of cosine

e.12-3 In Exercise *e*.9-14 we sampled a sinusoidal signal at its Nyquist rate. Let us consider this exercise again, but now for the general case, i.e., $x(t) = A\cos(2\pi f_0 t + \theta)$, with amplitude A, frequency f_0 and phase θ (in [rad]). We take integer N samples of $x(t)$, with a sampling frequency f_s equal to the Nyquist rate of $x(t)$, and with a measurement time T. A DFT is computed for this signal, resulting in coefficients X_k, for $k = 0, 1, \ldots, N-1$.

(a) Prove that when no effects of leakage are visible in the DFT, N must be an even number.

(b) For N even, compute the DFT coefficients X_k of $x(t)$ for any A, f_0 and $\theta \in \mathbb{R}$.

e| 12.c Computer exercises

e.12-4 From Example 5.1, we know that the (continuous-time) pulse function has the sinc function as its Fourier transform. Specifically, $x(t) = \Pi\left(\frac{t}{\tau}\right)$ and $X(f) = \tau\,\mathrm{sinc}(\tau f)$ are a Fourier transform pair, (5.11).

In this exercise, we explore the discrete-time counterpart of this Fourier transform pair. The Discrete Fourier Transform (DFT) is applied to discrete-time sequence x_n and we focus on interpreting the result of the DFT in the frequency domain.

Consider a unit pulse function $x(t)$ with pulse duration $\tau = 4$ s. The function is sampled with sampling interval $\Delta t = 1$ s, for time duration $T = 20$ s, resulting in discrete-time sequence x_n. Assume that sampling begins before and ends after the pulse, such that the pulse is centered within sequence x_n.

(a) Create and plot discrete-time sequence x_n.

(b) Compute the DFT of x_n.
Hint: in Python, one can apply the Fast Fourier Transform (FFT), an extremely efficient algorithmic implementation of the DFT, through the function `numpy.fft.fft`.

(c) Plot the modulus of the DFT of x_n, i.e., magnitude spectrum $|X_k|$.

The DFT is computed from 0 Hz, for positive frequencies up to $\frac{f_s}{2}$ Hz, in steps of $\Delta f = \frac{1}{T}$ Hz, after which the negative frequencies follow from $-\frac{f_s}{2}$ to $-\Delta f$ Hz. For a convenient visualization, we may want to shift (the coefficients of) the negative frequencies as to cover the interval $[-\frac{f_s}{2}, \frac{f_s}{2})$, symmetric about $f = 0$.

(d) Shift (the coefficients of) the negative frequencies to the left-hand side to create a plot of magnitude spectrum $|X_k|$, symmetric about $f = 0$. Restore the frequency axis by multiplying index k by the frequency step size Δf.
Hint: in Python, one can use the function `numpy.fft.fftshift` to shift (the coefficients of) the negative frequencies.

e.12-5

In this exercise, we explore the Dirac delta function in discrete time. We know that the Dirac delta function is defined to be zero everywhere, except for $t = 0$: $\delta(t) = 0, \ t \neq 0,$ (B.8), with unit area: $\int_{-\infty}^{\infty} \delta(t)\,\mathrm{d}t = 1,$ (B.9). Its value at $t = 0$ s is infinity, and its width is zero. In discrete time, this means that it is zero at all samples, except for one sample, which has a (large) non-zero value.

Consider signal
$$x(t) = \delta(t).$$

Signal $x(t)$ is sampled with sampling interval $\Delta t = 0.001$ s, starting at $t = 0.000$ s, for time duration T, such that a total of $N = 256$ samples are obtained. As a result, all samples of discrete-time sequence x_n have a value of 0, except for the first sample, which has a value equal to the sampling frequency f_s, due to the Dirac pulse occurring at $t = 0$ s. A discrete sample represents a time interval of $\Delta t = \frac{1}{f_s}$, and by having the amplitude equal to f_s, we ensure that the area of the Dirac pulse equals 1.

(a) Create and plot discrete-time sequence x_n. Show the samples as a function of the 'restored' continuous time t in [s] on the horizontal axis (hereafter referred to as 'time t').

(b) Compute the DFT of x_n.

(c) Plot magnitude spectrum $|X_k|$ as a function of the 'restored' continuous frequency f in [Hz] on the horizontal axis (hereafter referred to as 'frequency f'), covering the frequency range $[0, f_s)$. Comment on its appearance.

(d) Plot phase spectrum $\angle X_k$ as a function of frequency f, covering the frequency range $[0, f_s)$. Comment on its appearance.
Hint: in Python, one can apply the function `numpy.angle` to the DFT result.

e.12-6

This exercise is a follow-up to Exercise e.12-5, in which we explored the Dirac delta function in discrete time.

Consider again a signal $x(t)$, which is sampled with sampling interval $\Delta t = 0.001$ s, starting at $t = 0.000$ s, for time duration T, such that a total of $N = 256$ samples are obtained. All samples have a value of 0, except for the *second* sample, which has a value equal to f_s. Hence, the Dirac delta function is delayed by $t_0 = \Delta t = 0.001$ s compared to Exercise e.12-5.

(a) Create and plot discrete-time sequence x_n. Show the samples as a function of time t.

(b) Compute the DFT of x_n and plot amplitude spectrum $|X_k|$ and phase spectrum $\angle X_k$ as a function of frequency f, covering the frequency range $[0, f_s)$.

The time shift Fourier transform theorem, (6.2), states that when we delay the signal $x(t) = \delta(t)$ by an amount t_0, the Fourier transform becomes:
$$\int_{-\infty}^{\infty} \delta(t - t_0) e^{-j2\pi f t}\,\mathrm{d}t = e^{-j2\pi f t_0}$$
Hence, the spectrum has a phase of $-2\pi f t_0$, while the amplitude spectrum remains 1, i.e., the same. When you calculate the phase of a signal, you can only determine the phase modulus 2π. For instance, Python will return, using the four-quadrant arctan function, angles between $-\pi$ and $+\pi$. What you then often get to see, is a kind of 'sawtooth' shape. There is a way to undo this so-called wrapping, and that is appropriately called phase unwrapping.

(c) Apply unwrapping to the phase of the shifted Dirac delta function and plot the unwrapped phase spectrum $\angle X_k$ as a function of frequency f, covering the frequency range $[0, f_s)$.
Hint: in Python, one can apply the function `numpy.unwrap` to the result obtained using `numpy.angle`.

(d) Now create and plot a discrete-time sequence y_n, similar to x_n, but with an even more delayed Dirac delta function, i.e., create the Dirac delta function at the *eleventh* (instead of the second) time sample. Hence, the Dirac delta function is delayed by $t_0 = 10\Delta t = 0.010$ s compared to the Dirac delta function in Exercise *e*.12-5.

(e) Compute the DFT of y_n and plot amplitude spectrum $|Y_k|$ and phase spectrum $\angle Y_k$ as a function of frequency f, covering the frequency range $[0, f_s)$. Plot the latter without and with phase unwrapping.

e.12-7 This exercise is a follow-up to Exercise *e*.12-4, in which unit pulse function $x(t)$ with pulse duration $\tau = 4$ s was sampled with sampling interval $\Delta t = 1$ s, for time duration $T = 20$ s, resulting in discrete-time sequence x_n.

(a) Zero-pad sequence x_n from $T = 20$ s to $T_y = 100$ s, yielding sequence y_n, and compute the DFT of y_n.

Hint: in Python, instead of first zero-padding sequence x_n to create sequence y_n and then applying the function `numpy.fft.fft` to sequence y_n, one can also use the original sequence x_n as the first argument of `numpy.fft.fft` and set the second argument of `numpy.fft.fft` to 100 (in this specific case), which means that the original 20-sample signal is padded with 80 zero samples to create a signal of in total 100 samples before the DFT (FFT) is applied.

(b) Plot magnitude spectrum $|Y_k|$.

(c) Shift (the coefficients of) the negative frequencies to the left-hand side to create a plot of magnitude spectrum $|Y_k|$, symmetric about $f = 0$. Restore the frequency axis by multiplying index k by the frequency step size Δf.

e.12-8 Consider the cosine signal

$$x(t) = \cos(2\pi f_0 t)$$

with a frequency of $f_0 = 4.3$ Hz, and unit amplitude.

(a) Plot continuous-time signal $x(t)$.

(b) Compute and plot Fourier transform $X(f)$.

(c) Apply a (rectangular) measurement window of $T = 1$ s, symmetric about $t = 0$, to signal $x(t)$, and plot the result in the time domain, i.e., time-windowed signal $x_w(t)$.

(d) Compute the Fourier transform of the time-windowed signal, $X_w(f)$, and plot magnitude spectrum $|X_w(f)|$.

(e) Sample signal $x_w(t)$ with sampling frequency $f_s = 64$ Hz, and plot the time-windowed and sampled signal, i.e., discrete-time sequence x_n. Show the samples as a function of time t.

(f) Compute the DFT of x_n and plot magnitude spectrum $|X_k|$ as a function of frequency f, symmetric about $f = 0$.

Hint 1: multiply Python's FFT result by Δt, cf. (12.5) and (12.7), and divide by T, cf. (8.3), to maintain the analogy with the Fourier transform of infinite-duration, continuous-time signal $x(t)$.

Hint 2: in Python, to create a frequency vector, one can use the function `numpy.fft.fftfreq(N, 1/fs)`, where `N` is the number of samples and `fs` is the sampling frequency f_s; the second argument therefore equals $\frac{1}{f_s} = \Delta t$. This function returns a frequency vector of length N that starts at 0 Hz, increases in steps of $\Delta f = \frac{1}{T}$ Hz up to $\frac{f_s}{2}$ Hz for the positive frequencies, and then continues with the negative frequencies from $-\frac{f_s}{2}$ to $-\Delta f$ Hz. Note that this ordering exactly matches that of the DFT coefficient sequence X_k. Consequently, when wishing to plot the frequency range from $-\frac{f_s}{2}$ to $\frac{f_s}{2}$ Hz, i.e., a spectrum symmetric about $f = 0$, one can create a frequency vector using `numpy.fft.fftfreq`, in which case it is not necessary to shift the negative-frequency coefficients of the DFT coefficient sequence X_k using `numpy.fft.fftshift`. The DFT result can be directly plotted as a function of the created frequency vector.

(g) Zero-pad sequence x_n with $M = 64$ zero samples, yielding sequence y_n. Separately, zero-pad sequence x_n with $M = 192$ zero samples, yielding sequence z_n. Compute the DFT of both y_n and z_n, and plot magnitude spectra $|Y_k|$ and $|Z_k|$ as a function of frequency f, symmetric about $f = 0$.

(h) Comment on the effect of zero-padding on the outcome of the DFT by comparing magnitude spectra $|X_k|$, $|Y_k|$ and $|Z_k|$.

e.12-9

This exercise is a follow-up to Exercise *e*.12-8. Again consider the cosine signal

$$x(t) = \cos(2\pi f_0 t)$$

with a frequency of $f_0 = 4.3$ Hz, and unit amplitude.

(a) Apply a rectangular measurement window of $T = 1$ s, symmetric about $t = 0$, to signal $x(t)$, sample time-windowed signal $x_{w,\text{rect}}(t)$ with sampling frequency $f_s = 64$ Hz, and plot the time-windowed and sampled signal, i.e., discrete-time sequence $x_{n,\text{rect}}$. Show the samples as a function of time t.
Note: this is a repetition of parts (c) and (e) of Exercise *e*.12-8.

(b) Apply a raised cosine measurement window of $T = 1$ s, symmetric about $t = 0$, to signal $x(t)$, sample time-windowed signal $x_{w,\text{rcos}}(t)$ with sampling frequency $f_s = 64$ Hz, and plot the time-windowed and sampled signal, i.e., discrete-time sequence $x_{n,\text{rcos}}$. Show the samples as a function of time t.

(c) Compute the DFT of both $x_{n,\text{rect}}$ and $x_{n,\text{rcos}}$, and plot magnitude spectra $|X_{k,\text{rect}}|$ and $|X_{k,\text{rcos}}|$ as a function of frequency f, symmetric about $f = 0$.
Note: this is *in part* a repetition of Exercise *e*.12-8(f).

(d) Comment on the effects of the two windows on the outcome of the DFT by comparing magnitude spectra $|X_{k,\text{rect}}|$ and $|X_{k,\text{rcos}}|$. Comment specifically on the phenomenon of spectral leakage.

Solutions

s| 12.2 Definition of the DFT and inverse DFT

s.12-1

The X_k coefficients can be obtained by substituting $f = k\Delta f = k\frac{1}{N\Delta t}$ in the expression for the DTFT of an infinite-duration (shifted) pulse, the third pair in Table G.1:

$$X_k = X_{\Delta t}(f = k\frac{1}{N\Delta t}) = \Delta t\,\frac{\sin\left[2\pi\frac{k}{N}\left(\frac{M}{2}\right)\right]}{\sin\left[\frac{2\pi\frac{k}{N}}{2}\right]}e^{-j2\pi\frac{k}{N}\left(\frac{M-1}{2}\right)} = \Delta t\,\frac{\sin\left[k\frac{2\pi}{N}\left(\frac{M}{2}\right)\right]}{\sin\left[\frac{k\frac{2\pi}{N}}{2}\right]}e^{-jk\frac{2\pi}{N}\left(\frac{M-1}{2}\right)}$$

The proof is very similar to that of Exercise *e*.11-1. Substitute x_n in the expression for the DFT (12.5):

$$X_k = \Delta t\sum_{n=0}^{N-1}x_n e^{-jk\frac{2\pi}{N}n} = \Delta t\sum_{n=0}^{M-1}e^{-jk\frac{2\pi}{N}n} = \Delta t\sum_{n=0}^{M-1}\left(e^{-jk\frac{2\pi}{N}}\right)^n$$

On the right-hand side we obtain the geometric series identity, (A.22), with $a = e^{-jk\frac{2\pi}{N}}$, $k = n$ and $N = M$. The identity states that we need to consider the cases $a = 1$ and $a \neq 1$. Here, for $k = 0$ we get $a = e^0 = 1$ so this case is considered first. For $k = 0$, $X_0 = \Delta t M$.

For $k \neq 0$ we obtain: $X_k = \Delta t\left(\frac{1-e^{-jk\frac{2\pi}{N}M}}{1-e^{-jk\frac{2\pi}{N}}}\right)$ (*)

The denominator can be written as $e^{-jk\frac{\pi}{N}}\left(e^{jk\frac{\pi}{N}} - e^{-jk\frac{\pi}{N}}\right) = e^{-jk\frac{\pi}{N}}2j\sin\left[k\frac{2\pi}{N}\frac{1}{2}\right]$, using (C.3).

Similarly, the numerator can be written as $e^{-jk\frac{\pi}{N}M}2j\sin\left[k\frac{2\pi}{N}\frac{M}{2}\right]$. Substitute in (*):

$$X_k = \Delta t\,\frac{e^{-jk\frac{\pi}{N}M}\;2j\sin\left[k\frac{2\pi}{N}\frac{M}{2}\right]}{e^{-jk\frac{\pi}{N}}\;2j\sin\left[k\frac{2\pi}{N}\frac{1}{2}\right]} = \Delta t\,e^{-jk\frac{2\pi}{N}\left(\frac{M-1}{2}\right)}\frac{\sin\left[k\frac{2\pi}{N}\left(\frac{M}{2}\right)\right]}{\sin\left[\frac{k\frac{2\pi}{N}}{2}\right]}$$ **qed**

X_k is a complex-valued function. For $k = 0$ this expression for X_k results in a $\frac{0}{0}$ case; using L'Hôpital's rule yields $X_0 = \Delta t M$, as found above.
When $M = N$ we obtain the fourth pair in Table H.1:

$$X_k = \Delta t\,e^{-jk\frac{2\pi}{N}\left(\frac{N-1}{2}\right)}\frac{\sin\left[k\frac{2\pi}{N}\left(\frac{N}{2}\right)\right]}{\sin\left[\frac{k\frac{2\pi}{N}}{2}\right]} = \Delta t\,e^{-jk\frac{2\pi}{N}\left(\frac{N-1}{2}\right)}\frac{\sin[k\pi]}{\sin\left[\frac{k\frac{2\pi}{N}}{2}\right]} \qquad (**)$$

where the ratio of sine functions on the right-hand side of (**) is zero for all k (because $\sin[k\pi] = 0\,\forall\,k$), except for $k = 0$. For $k = 0$, we get another $\frac{0}{0}$ case; using L'Hôpital's rule yields $X_0 = \Delta t N$.

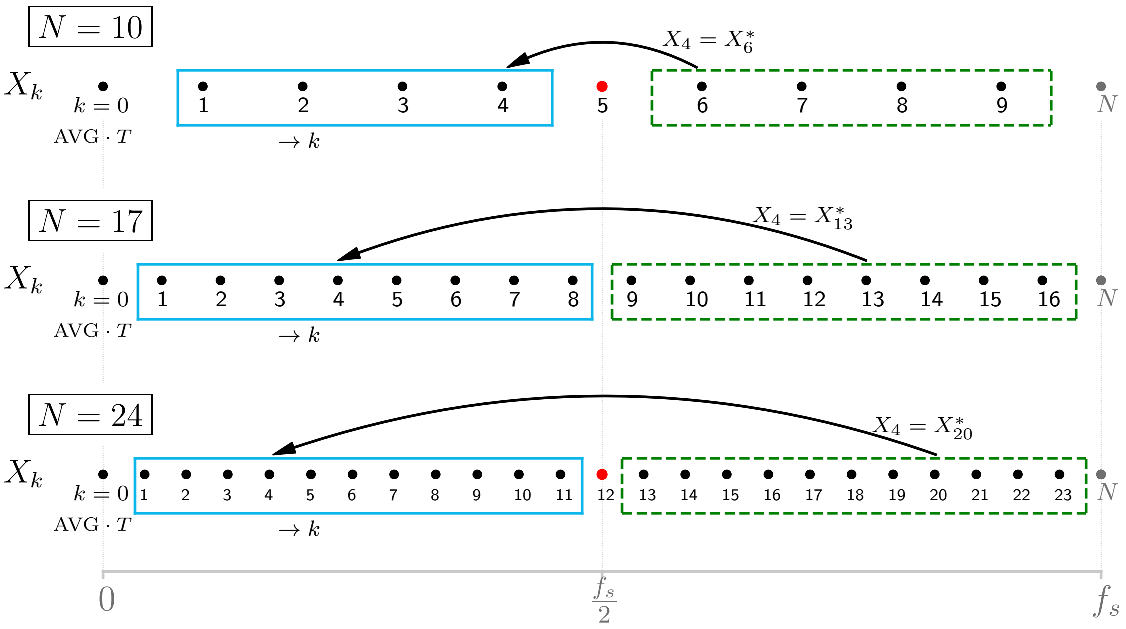
We conclude, that when $M = N$: $X_k = \begin{cases}\Delta t N & k = 0\\ 0 & \text{else}\end{cases}$

Expressed differently: $X_k = \Delta t N\,\delta_k$, with δ_k the Kronecker delta, see Table H.1.

s | 12.3 DFT frequencies, properties, and diagram

s.12-2

The DFT diagram for the three cases is illustrated below.



In all cases, the average of the samples (AVG) can be computed through dividing the coefficient at $k = 0$, X_0, by the measurement time T. In all cases, the diagram shows in gray the continuous frequency interval $[0, f_s)$.

(a) For $N = 10$, an even number of samples, the DFT results in an even number of coefficients, $k = 0, 1, \ldots, 9$. We obtain $\frac{N}{2} - 1 = 4$ positive frequencies (blue rectangle) and 4 negative (green dashed rectangle) frequencies. The coefficient at Nyquist appears at $k = \frac{N}{2} = 5$ (red dot). The coefficient X_4 ($k = 4$) is the complex conjugate of the coefficient X_6 ($k = N - 4 = 10 - 4 = 6$): $X_4 = X_6^*$, or, equivalently $X_6 = X_4^*$.

(b) For $N = 17$, an odd number of samples, the DFT results in an odd number of coefficients, $k = 0, 1, \ldots, 16$. We obtain $\frac{N-1}{2} = 8$ positive frequencies and 8 negative frequencies. Because of the odd number of coefficients, the DFT does not result in a coefficient at the Nyquist frequency. The coefficient X_4 is the complex conjugate of the coefficient X_{13}.

(c) For $N = 24$, an even number of samples, the DFT results in an even number of coefficients. We obtain $\frac{N}{2} - 1 = 11$ positive frequencies and 11 negative frequencies. The coefficient at Nyquist appears at $k = \frac{N}{2} = 12$. The coefficient X_4 is the complex conjugate of the coefficient X_{20}.

s | 12.5 Two more examples: DFT of cosine

s.12-3

(a) When the DFT shows no effects of leakage, the frequency of the cosine, f_0, must be an integer multiple $\ell \in \mathbb{N}^+$ of the DFT frequency step $\Delta f = \frac{1}{T}$ (see Example 12.5). Then: $f_0 = \ell \Delta f = \frac{\ell}{T}$. We sample with a sampling frequency f_s equal to the Nyquist rate, so $f_s = 2 f_0$. The number of samples N equals $T f_s = \frac{\ell}{f_0} 2 f_0 = 2\ell$. Hence, $\forall\, \ell$, N is even.

(b) Exercise *e*.9-14 showed that, when sampling a cosine at its Nyquist rate, $f_s = 2f_0$, the samples obtained will be: $x_n = A\cos\theta(-1)^n$ (see Solution *s*.9-14). The $\frac{N}{2}$ even-indexed samples $x_0, x_2, \dots, x_{N-2}$ are all equal to $A\cos\theta$, the $\frac{N}{2}$ odd-indexed samples $x_1, x_3, \dots, x_{N-1}$ all equal $-A\cos\theta$. As explained on page 130 of the companion book, Theory, the coefficient at the Nyquist frequency is a real number equal to the sum of the even-indexed samples minus the sum of the odd-indexed samples, multiplied by Δt. We get:

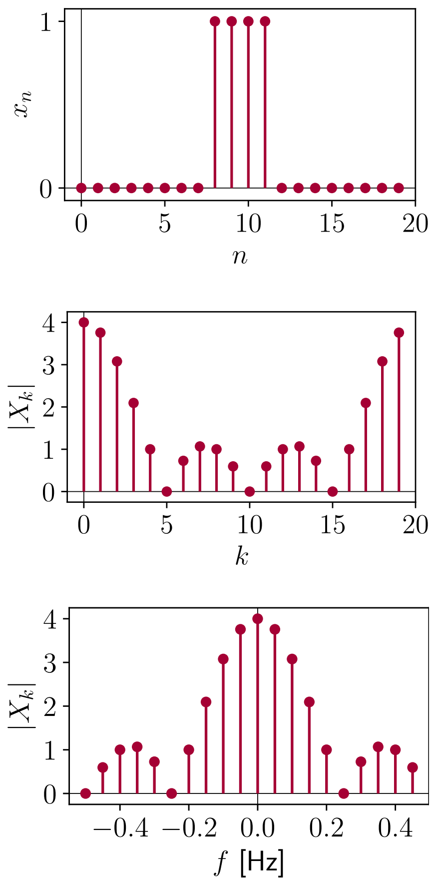
$$\begin{aligned} X_{k=\frac{N}{2}} &= \Delta t\left((x_0 + x_2 + \cdots + x_{N-2}) - (x_1 + x_3 + \cdots + x_{N-1})\right) \\ &= \Delta t\left(\frac{N}{2}A\cos\theta - \frac{N}{2}(-A)\cos\theta\right) = \Delta t N A\cos\theta \end{aligned}$$

All other DFT coefficients are zero. The DFT spectrum is real, even, and has just one peak at $k = \frac{N}{2}$.

s | 12.c Computer exercises

s.12-4

(a) From $T = 20$ s and $\Delta t = 1$ s, we know that there are $N = 20$ samples. The top figure on the right shows unit pulse function $x(t) = \Pi\left(\frac{t}{\tau}\right)$ with width $\tau = 4$ s and shifted to time $t = 10$ s, in discrete time, hence, as sequence x_n.



(b)-(c) The Fourier transform is evaluated at $N = 20$ discrete frequencies. The middle figure on the right shows the modulus of the DFT, i.e., magnitude spectrum $|X_k|$, of discrete-time sequence x_n, with frequency index k running from 0 to 19. The (magnitude) spectrum of the sampled signal is periodic in sampling frequency f_s, which equals 1 Hz here. Therefore, it is computed for just one period $[0, f_s)$. Spectra are commonly presented and interpreted as *double-sided*, so in this figure we can interpret the spectral components with indices $k = 10$ to 19 as the components corresponding to the *negative* frequencies, analogous to Example 12.3.

We can recognize a bit of a sinc function. The DFT is computed from 0 Hz, for positive frequencies up to $\frac{f_s}{2} = 0.5$ Hz, in steps of $\Delta f = \frac{1}{T} = 0.05$ Hz, after which the negative frequencies follow from -0.5 to -0.05 Hz. $X(f) = 4\,\mathrm{sinc}(4f)$ has its first null at $f = 0.25$ Hz, hence, here at $k = 5$.

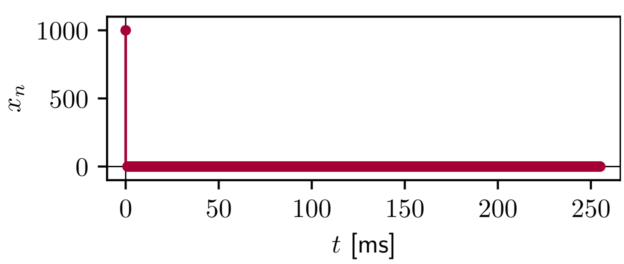
(d) The bottom figure on the right shows magnitude spectrum $|X_k|$ as a double-sided spectrum, symmetric about $f = 0$ Hz, with frequency step size $\Delta f = \frac{1}{T} = 0.05$ Hz. The zero-frequency component now appears in the center of the spectrum, and the frequency range is from $-\frac{f_s}{2}$ to $\frac{f_s}{2}$, i.e., $[-\frac{f_s}{2}, \frac{f_s}{2})$. The shape of the sinc function can now be clearly seen and related to its continuous-time counterpart.

Basic Python code:

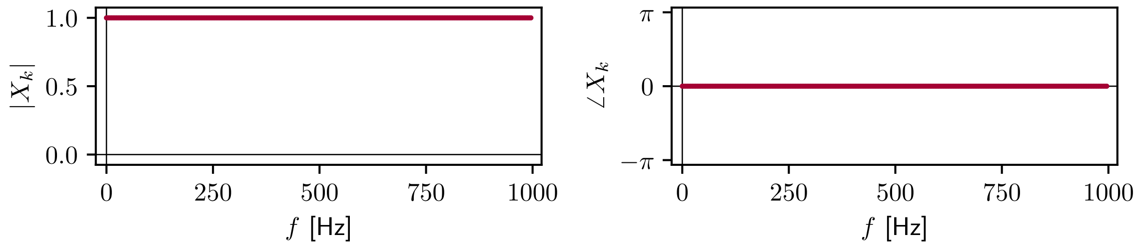
```
import numpy as np
from matplotlib import pyplot as plt

burgundy = [165/255,0/255,52/255]         # burgundy color

# create and plot unit pulse function x(t) as discrete-time sequence x_n
dt  = 1                                   # sampling interval Delta_t = 1 s
T   = 20                                  # signal duration T = 20 s
tau = 4                                   # pulse duration tau = 4 s
N   = int(T/dt)                           # number of samples
n   = np.arange(0, N)
xn  = np.zeros(N)                         # discrete-time sequence x_n
xn[(N-tau)//2:(N-tau)//2+tau] = 1         # pulse

plt.figure()
plt.subplot(3,2,1)                        # discrete-time sequence x_n
plt.stem(n, xn, basefmt=' ', linefmt=burgundy, markerfmt=burgundy)
plt.xlabel('$n$'); plt.ylabel('$x_n$')

# compute and plot (the modulus of) the DFT
Xk = dt * np.fft.fft(xn)     # DFT; multiply Python's FFT result by Delta_t

plt.subplot(3,2,3)                        # modulus of DFT
plt.stem(n, np.abs(Xk), basefmt=' ', linefmt=burgundy, markerfmt=burgundy)
plt.xlabel('$k$'); plt.ylabel('$|X_k|$')

# shift the negative frequencies to the left-hand side such that
# the zero-frequency component appears in the center of the spectrum
# and plot (the modulus of) the DFT with restored frequency axis
df       = 1/T                            # frequency step size [Hz]
flim     = int(N/2)
f_vec    = np.arange(-flim, flim) * df    # frequency axis
Xk_shift = np.fft.fftshift(Xk)            # shifted DFT

plt.subplot(3,2,5)                        # modulus of shifted DFT
plt.stem(f_vec, np.abs(Xk_shift), basefmt=' ', linefmt=burgundy, markerfmt
    =burgundy)
plt.xlabel('$f$ [Hz]'); plt.ylabel('$|X_k|$')
```

S.12-5

(a) The figure on the right shows sequence x_n. It might not be directly clear from the figure, as the samples are plotted so close to one another, but it actually is a discrete-time sequence.



(b)-(d) On the next page, the magnitude and phase spectra, $|X_k|$ and $\angle X_k$, respectively, are shown. Again, it might not be directly clear, due to the overlap, but the DFT result is shown as dots, representing the discrete-frequency sequence X_k. The amplitude $|X_k|$ equals 1, as expected, since the Fourier transform of a Dirac delta function equals 1 (remember the sifting property of the Dirac delta function, (B.12)): $\int_{-\infty}^{\infty} \delta(t) e^{-j2\pi f t}\,\mathrm{d}t = 1$. Remember also Fourier transform pair $\delta(t) \overset{\mathcal{F}}{\leftrightarrow} 1$, (5.14). The phase $\angle X_k$ equals zero, as expected, due to the Dirac pulse occurring at time $t = 0$ s.

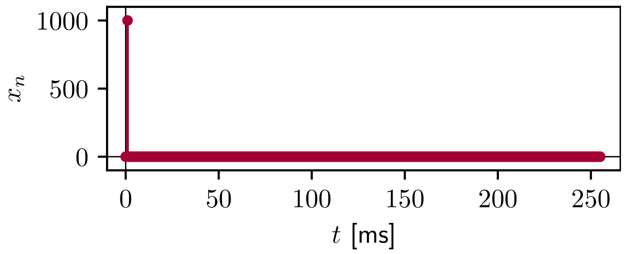


Basic Python code:

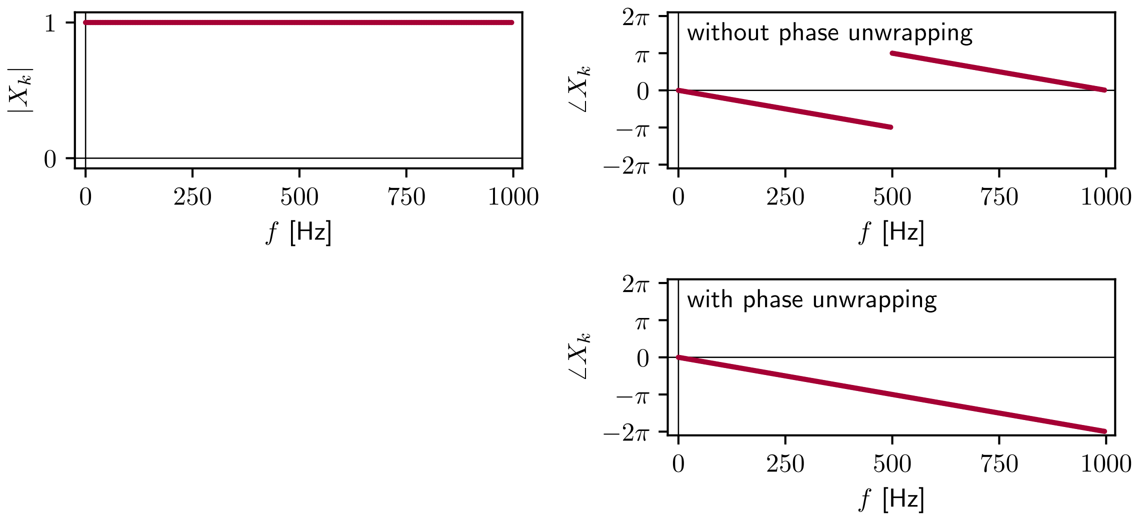
```
import numpy as np
from matplotlib import pyplot as plt

burgundy = [165/255,0/255,52/255]     # burgundy color

# define required parameters
dt     = 0.001                         # time interval or time step size [s]
N      = 256                           # number of samples
Tmeas = N * dt                         # total measurement time [s]
df     = 1 / Tmeas                     # frequency step size [Hz]
fs     = 1 / dt                        # sampling frequency [Hz]

# create discrete-time sequence x_n and time vector
xn        = np.zeros(N)                # zero signal
xn[0]     = fs                         # Dirac pulse at t = 0.000 s
time_sec = np.arange(0, Tmeas, dt)     # time vector [s]
time_ms  = time_sec * 1e3              # time vector [ms]

# plot discrete-time sequence x_n as function of restored time axis
plt.figure()
plt.stem(time_ms, xn, basefmt=' ', linefmt=burgundy, markerfmt=burgundy)
plt.xlabel('$t$ [ms]'); plt.ylabel('$x_n$')

# DFT of sampled signal x_n
# no division by Tmeas needed since Dirac pulse has only 1 non-zero value
Xk   = dt * np.fft.fft(xn)    # DFT; multiply Python's FFT result by Delta_t
abs_Xk   = np.abs(Xk)                  # amplitude
phase_Xk = np.angle(Xk)                # phase
f_vec    = np.arange(0, df * N, df)    # frequency axis

# plot amplitude spectrum and phase spectrum
plt.figure()
plt.subplot(3,2,1)                     # amplitude
plt.plot(f_vec, abs_Xk, 'o', color=burgundy, markersize=1)
plt.xlabel('$f$ [Hz]'); plt.ylabel('$|X_k|$')
plt.subplot(3,2,2)                     # phase
plt.plot(f_vec, phase_Xk, 'o', color=burgundy, markersize=1)
plt.xlabel('$f$ [Hz]'); plt.ylabel('$\\angle X_k$')
```

S.12-6

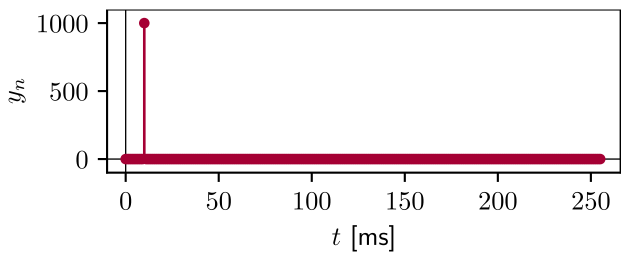
(a) The figure on the right shows sequence x_n. Similar to Solution *S*.12-5(a), it might not directly be clear from the figure, but it actually is a discrete-time sequence. The difference with the figure in Solution *S*.12-5(a) is difficult to see, as the Dirac pulse has shifted by only one time step, hence, one sample.



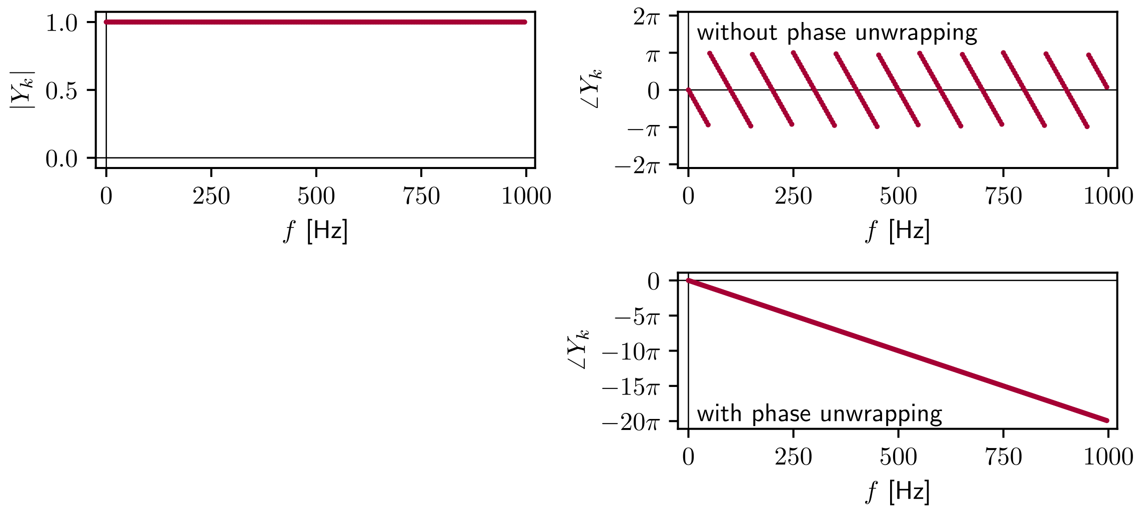
(b)-(c) Below, the amplitude and phase spectra, $|X_k|$ and $\angle X_k$, respectively, are shown. Again, it might not directly be clear, due to the overlap, but the DFT result is shown as dots, representing the discrete-frequency sequence X_k. As one can see on the left, amplitude spectrum $|X_k|$ has indeed not changed compared to when the Dirac pulse occurred at the first time instant, i.e., at $t = 0.000$ s (see Solution *S*.12-5(b)-(d)). However, phase spectrum $\angle X_k$ has. Below, the top figure on the right shows phase spectrum $\angle X_k$ without phase unwrapping. It indeed shows kind of a 'sawtooth' shape. The bottom figure on the right shows phase spectrum $\angle X_k$ with phase unwrapping. The phase angle is $-2\pi f t_0$ with $t_0 = 0.001$ s.



(d) The figure on the right shows discrete-time sequence y_n with a Dirac pulse at the eleventh time instant.



(e) On the next page, the amplitude and phase spectra, $|Y_k|$ and $\angle Y_k$, respectively, are shown. As one can see on the left, amplitude spectrum $|Y_k|$ has (again) not changed compared to when the Dirac pulse occurred at the first or the second time instant (compare to the plots of $|X_k|$ in Solution *S*.12-5(b)-(d) and (b)-(c) of this solution). However, phase spectrum $\angle Y_k$ has. The top figure on the right shows phase spectrum $\angle Y_k$ without phase unwrapping. A clear 'sawtooth' shape is visible. The bottom figure on the right shows phase spectrum $\angle Y_k$ with phase unwrapping. Note the difference in the vertical axis range between the phase spectra without and with phase unwrapping.



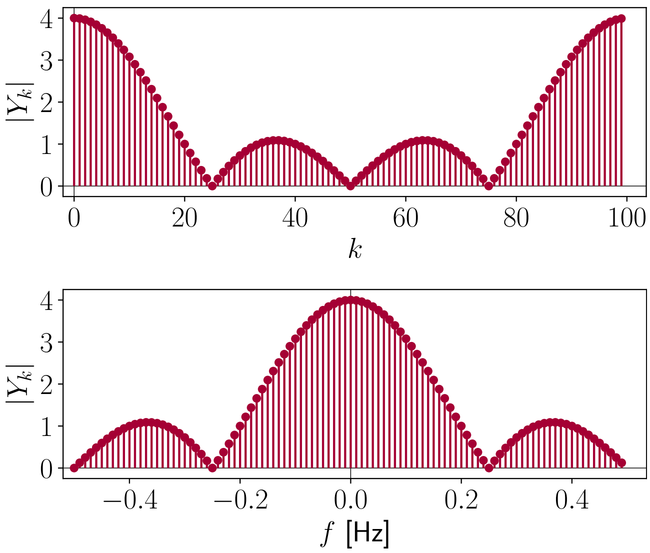
Basic Python code:

```
import numpy as np
from matplotlib import pyplot as plt

burgundy = [165/255,0/255,52/255]        # burgundy color

# define required parameters (equal to those in previous exercise)
dt = 0.001; N = 256; Tmeas = N * dt; df = 1 / Tmeas; fs = 1 / dt

# create discrete-time sequence x_n and time vector
xn2      = np.zeros(N)         # zero signal
xn2[1]   = fs                  # Dirac pulse at second time instant
time_sec = np.arange(0, Tmeas, dt); time_ms  = time_sec * 1e3

# plot discrete-time sequence x_n as function of restored time axis
plt.figure()
plt.stem(time_ms, xn2, basefmt=' ', linefmt=burgundy, markerfmt=burgundy)
plt.xlabel('$t$ [ms]'); plt.ylabel('$x_n$')

# DFT of sampled signal x_n
# no division by Tmeas needed since Dirac pulse has only 1 non-zero value
Xk2 = dt * np.fft.fft(xn2)   # DFT; multiply Python's FFT result by Delta_t
abs_Xk2          = np.abs(Xk2)                           # amplitude
phase_Xk2        = np.angle(Xk2)                         # phase
phaseunwrap_Xk2 = np.unwrap(phase_Xk2)                   # unwrapped phase
f_vec            = np.arange(0, df * N, df)              # frequency axis

# plot amplitude spectrum and phase spectrum
plt.figure()
plt.subplot(3,2,1)
plt.plot(f_vec, abs_Xk2, 'o', color=burgundy, markersize=1)
plt.xlabel('$f$ [Hz]'); plt.ylabel('$|X_k|$')
plt.subplot(3,2,2)                                  # without phase unwrapping
plt.plot(f_vec, phase_Xk2, 'o', color=burgundy, markersize=1)
plt.xlabel('$f$ [Hz]'); plt.ylabel('$\\angle X_k$')
plt.subplot(3,2,4)                                  # with phase unwrapping
plt.plot(f_vec, phaseunwrap_Xk2, 'o', color=burgundy, markersize=1)
plt.xlabel('$f$ [Hz]'); plt.ylabel('$\\angle X_k$')
```

```
# the remainder of the code for (d) and (e) is highly similar to the code
# above for (a)-(c) and is therefore omitted to avoid excessive repetition
```

s.12-7

(a)-(b) From $T_y = 100$ s and $\Delta t = 1$ s, we know that sequence y_n consists of $N_y = 100$ samples. The Fourier transform is evaluated at $N_y = 100$ discrete frequencies. The top figure on the right shows magnitude spectrum $|Y_k|$.



(c) The bottom figure on the right shows magnitude spectrum $|Y_k|$ as a double-sided spectrum, symmetric about $f = 0$ Hz. Just as in Solution *s*.12-4, the frequency range still covers $[-\frac{f_s}{2}, \frac{f_s}{2})$, with $f_s = \frac{1}{\Delta t} = 1$ Hz, which is not influenced by zero-padding, but the frequency step size has now become $\Delta f_y = \frac{1}{T_y} = 0.01$ Hz. Compared to the bottom graph in Solution *s*.12-4, the graph here is much more dense. Previously, the frequency step size was $\Delta f = 0.05$ Hz, and now it is $\Delta f_y = 0.01$ Hz.

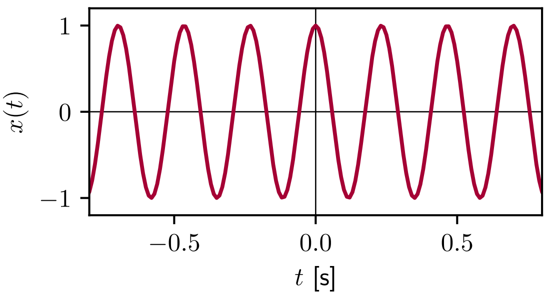
Basic Python code:

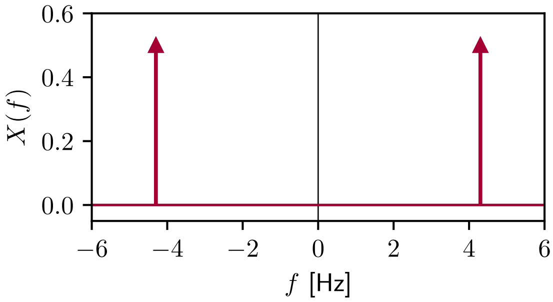
```
import numpy as np
from matplotlib import pyplot as plt

burgundy = [165/255,0/255,52/255]  # burgundy color

# create unit pulse function x(t) as discrete-time sequence x_n
# (identical to previous exercise)
dt  = 1; T = 20; tau = 4; N = int(T/dt); n = np.arange(0, N)
xn  = np.zeros(N); xn[(N-tau)//2:(N-tau)//2+tau] = 1

# compute and plot the modulus of the DFT with padding of 80 zero samples
Ty   = 100                # signal duration T = 100 s of zero-padded signal
Nzp  = int(Ty/dt)         # number of samples
nzp  = np.arange(0, Nzp)
Ykzp = dt * np.fft.fft(xn, Nzp) # DFT; multiply Python's FFT result by dt

plt.figure()
plt.subplot(2,1,1)
plt.stem(nzp, np.abs(Ykzp), basefmt=' ', linefmt=burgundy, markerfmt=
    burgundy)
plt.xlabel('$k$'); plt.ylabel('$|Y_k|$')

# shift the negative frequencies to the left-hand side such that
# the zero-frequency component appears in the center of the spectrum
# and plot (the modulus of) the DFT with restored frequency axis
```

```
dfzp        = 1/Ty                              # frequency step size [Hz]
flimzp      = int(Nzp/2)
f_veczp     = np.arange(-flimzp, flimzp) * dfzp  # frequency axis
Ykzp_shift = np.fft.fftshift(Ykzp)              # shifted DFT

plt.subplot(2,1,2)
plt.stem(f_veczp, np.abs(Ykzp_shift), basefmt=' ', linefmt=burgundy,
    markerfmt=burgundy)
plt.xlabel('$f$ [Hz]'); plt.ylabel('$|Y_k|$')
```

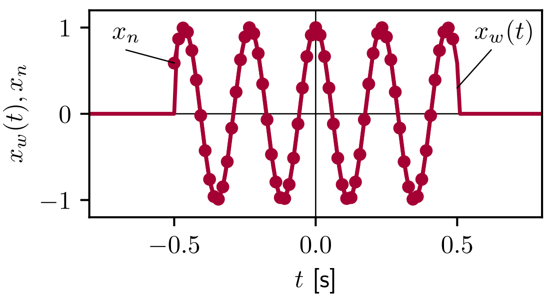
S.12-8

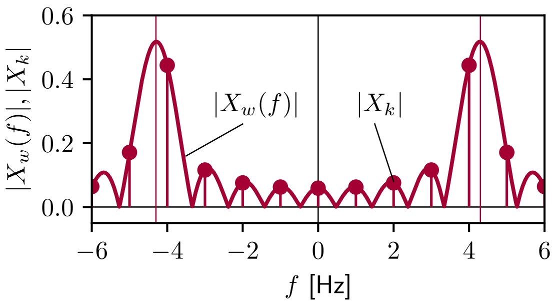
(a)-(b) In the left figure below, signal $x(t) = \cos(2\pi f_0 t)$ with $f_0 = 4.3$ Hz is shown as a continuous-time signal. At right, the corresponding Fourier transform in-the-limit, $X(f) = \frac{1}{2}(\delta(f+4.3) + \delta(f-4.3))$, (5.16), consisting of two Dirac pulses, is shown.





(c)-(f) In the left figure below, the rectangular time window, $w(t) = \Pi\left(\frac{t}{T}\right)$ with $T = 1$ s, has been applied according to (8.1). Time-windowed signal $x_w(t)$ is shown as a continuous curve. The time-windowed and sampled signal, i.e., discrete-time sequence x_n, is shown by the dots (remember: $f_0 = 4.3$ Hz and $f_s = 64$ Hz, hence, aliasing is by far no issue here, as $f_s \gg f_0$).

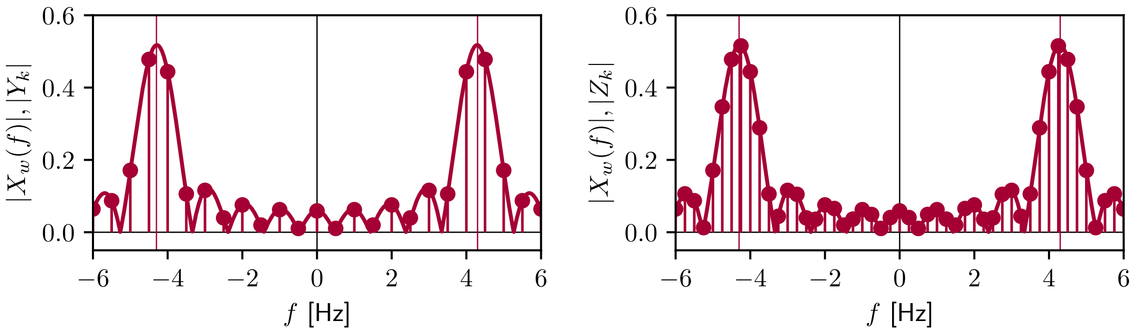




The right figure above shows, as a continuous curve, the magnitude of Fourier transform $X_w(f) = \frac{T}{2}\big(\mathrm{sinc}(T(f+f_0)) + \mathrm{sinc}(T(f-f_0))\big)$, see (8.2) and (8.3), of time-windowed signal $x_w(t)$, for continuous time and continuous frequency. In this case, $T = 1$ s, and thereby, the value of $X_w(f_0)$ equals the weight of the Dirac pulse of $X(f)$ at $f = f_0$, and equally at $f = -f_0$. Only a part of the (double-sided) spectrum is shown. The DFT has been applied to the $N = 64$ samples, i.e., discrete-time sequence x_n. Magnitude spectrum $|X_k|$ as obtained with the DFT is visualized by the stems in the figure. The FFT result has been multiplied, in terms of amplitude, by Δt, (12.5), and divided by T, the latter to account for the window length, see Section 8.2. The DFT result presents $X_w(f)$ evaluated at N discrete frequencies in the interval $[-\frac{f_s}{2}, \frac{f_s}{2})$, assuming N even. The figure here shows the spectrum only for frequencies of interest, i.e., $f \in [-6, 6]$ Hz.

The frequency step of the DFT is $\Delta f = \frac{1}{T}$, in this case 1 Hz. In the stem plot at right, we clearly see the effect of spectral leakage (cosine frequency f_0 is not an integer multiple of frequency step Δf).

(g) The two figures below show the results of zero-padding. In both cases, $|X_w(f)|$ as a continuous function of frequency is shown as a continuous curve, and the zero-padded DFT results, $|Y_k|$ and $|Z_k|$, are shown as stems. At left, showing $|Y_k|$, the $N = 64$ samples have been extended by $M = N$ zeros, and the frequency step has consequently become $\Delta f = \frac{1}{2}$ Hz. At right, showing $|Z_k|$, the $N = 64$ samples have been extended by $M = 3N$ zeros, resulting in $\Delta f = \frac{1}{4}$ Hz.



(h) Zero-padding results in having the Fourier transform of the time-windowed signal, $X_w(f)$, available (i.e., being evaluated) at more discrete frequencies. Zero-padding is *not* of any use at all in an attempt to reduce or avoid spectral leakage! One could say that zero-padding is actually 'fooling' the DFT algorithm by feeding it a longer measurement record (though just containing additional zeros), making it believe that the frequency step should be based on this longer measurement record.

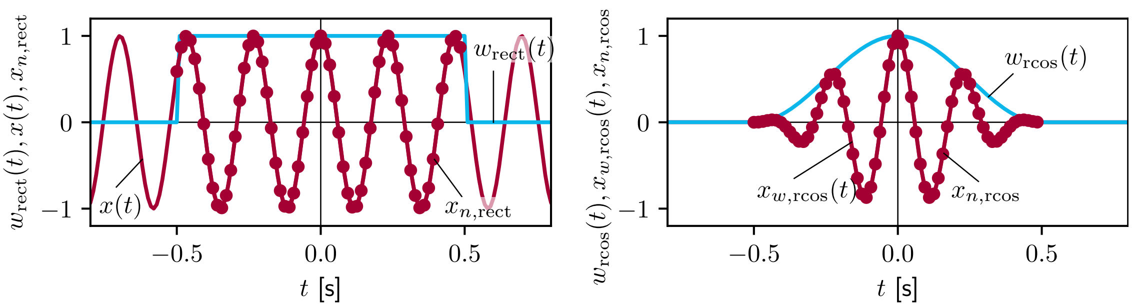
Basic Python code:

```
import numpy as np
from matplotlib import pyplot as plt

burgundy = [165/255,0/255,52/255]     # burgundy color

# demonstration using a simple cosine signal of leakage due to
# time-windowing (both in continuous time and discrete time)
# zero-padding does not help to reduce/avoid leakage
# (it just evaluates X_w(f) at more discrete frequencies)

# x(t) = cos(2 pi f0 t)            this is an even function (unit amplitude)
# X(f) = 1/2 delta(f+f0) + 1/2 delta(f-f0)    so, Fourier transform is real

# create cosine signal x(t) in continuous time using very small time step
f0 = 4.3                               # frequency [Hz]
t  = np.linspace(-2, 2, 401)           # time steps 0.01 s from -2 s to 2 s
xt = np.cos(2*np.pi*f0*t)              # cosine signal

plt.figure()
plt.subplot(3,2,1)                     # cosine function ('continuous time')
plt.plot(t, xt, color=burgundy)
plt.xlabel('$t$ [s]'); plt.ylabel('$x(t)$')
```

```
Xf  = ([0.5, 0.5])                        # Dirac pulses of X(f)
fd = ([-f0, f0])

plt.subplot(3,2,2)          # (analytical) Fourier transform (in-the-limit)
markerline, stemline, baseline = plt.stem(fd, Xf, basefmt=' ', linefmt=
    burgundy)
plt.setp(markerline, marker='^', color=burgundy)
plt.axhline(0, color=burgundy, linewidth=1)
plt.xlabel('$f$ [Hz]'); plt.ylabel('$X(f)$')

# apply (rectangular) time window w(t) = Pi(t/T), width T, centered at t=0
T  = 1                      # rectangular time window duration (Tmeas) [s]
fs = 64                     # sampling frequency [Hz] (discrete time)
N  = T*fs                   # number of samples

# unit amplitude pulse function width T and windowed signal x_w(t)
wt  = np.heaviside(t+T/2, 0) - np.heaviside(t-T/2, 0)
xwt = np.multiply(xt, wt)

# Fourier transform of continuous-time rectangular-windowed cosine
# X_w(f) is a continuous-frequency function
f   = np.linspace(-6, 6, 1201)# frequency steps 0.01 Hz from -6 Hz to 6 Hz
Xwf = T/2 * (np.sinc(T*(f+f0)) + np.sinc(T*(f-f0)))

# discrete-time (time-windowed and sampled) cosine signal x_n
t_vec = np.linspace(-T/2, T/2-1/fs, N)# time steps according to fs = 64 Hz
xn    = np.cos(2*np.pi*f0*t_vec)

# DFT of time-windowed and sampled signal x_n with amplitude scaling
# multiply Python's FFT result by Delta_t and divide by measurement time T
# to maintain equivalence with continuous-time signal x(t) (*Delta_t/T)
Xk    = np.fft.fft(xn) / N          # DFT; (*Delta_t/T equals divide by N)
f_vec = np.fft.fftfreq(N, 1/fs)     # frequency axis (discr freq) FFT output

# in the plots below, X_k is overlaid directly on X_w(f)
# (which matches as fs >> f0; 'aliases' due to sampling are far away)

plt.subplot(3,2,3)                  # windowed (and sampled) cosine signal
plt.plot(t, xwt, color=burgundy)
plt.plot(t_vec, xn, color=burgundy, marker='o', markersize=4)
plt.xlabel('$t$ [s]'); plt.ylabel('$x_w(t), x_n$')

plt.subplot(3,2,4)        # (D)Fourier transform rectangular-windowed signal
plt.axvline( f0, color=burgundy, linewidth=0.5)
plt.axvline(-f0, color=burgundy, linewidth=0.5)
plt.plot(f, np.abs(Xwf), color=burgundy)
plt.stem(f_vec, np.abs(Xk), basefmt=' ', linefmt=burgundy, markerfmt=
    burgundy)
plt.xlabel('$f$ [Hz]'); plt.ylabel('$|X_w(f)|,|X_k|$')

# zero-padding (explicit)

M = N                                           # number of zero-padding samples
yn_zp1    = np.concatenate((xn, np.zeros(M)))           # zero-padded signal
# DFT of time-windowed and sampled, zero-padded signal; amplitude scaling
```

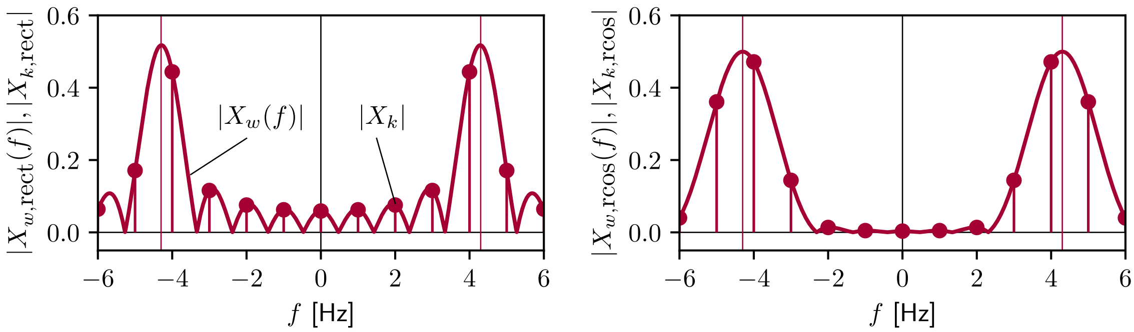
```
# (still by N, rather than N+M; zero-padding does not add samples,
# hence *Delta_t/T, with T the original non-zero-padded window length)
Yk_zp1     = np.fft.fft(yn_zp1) / N
# frequency array for (discrete freq) FFT output (after zero-padding)
f_vec_zp1 = np.fft.fftfreq(N+M, 1/fs)

zn_zp3    = np.concatenate((xn, np.zeros(3*M))) # longer zero-padded signal
# DFT of time-windowed and sampled, zero-padded signal; amplitude scaling
# (still by N, rather than N+3*M; zero-padding does not add samples)
Zk_zp3     = np.fft.fft(zn_zp3) / N
# frequency array for (discrete freq) FFT output (after zero-padding)
f_vec_zp3 = np.fft.fftfreq(N+(3*M), 1/fs)

plt.subplot(3,2,5)        # Fourier transform of zero-padded signal
plt.axvline( f0, color=burgundy, linewidth=0.5)
plt.axvline(-f0, color=burgundy, linewidth=0.5)
plt.plot(f, np.abs(Xwf), color=burgundy)
plt.stem(f_vec_zp1, np.abs(Yk_zp1), basefmt=' ', linefmt=burgundy,
    markerfmt=burgundy)
plt.xlabel('$f$ [Hz]'); plt.ylabel('$|X_w(f)|,|Y_k|$')

plt.subplot(3,2,6)        # Fourier transform of (longer) zero-padded signal
plt.axvline( f0, color=burgundy, linewidth=0.5)
plt.axvline(-f0, color=burgundy, linewidth=0.5)
plt.plot(f, np.abs(Xwf), color=burgundy)
plt.stem(f_vec_zp3, np.abs(Zk_zp3), basefmt=' ', linefmt=burgundy,
    markerfmt=burgundy)
plt.xlabel('$f$ [Hz]'); plt.ylabel('$|X_w(f)|,|Z_k|$')
```

(a) Since this (sub)exercise was a repetition of parts (c) and (e) of Exercise e.12-8, one can also have a look at Solutions s.12-8(a)-(b) and s.12-8(c)-(f). In the left figure below, continuous-time signal $x(t)$ is shown as a continuous burgundy curve and the rectangular time window, $w_{\text{rect}}(t) = \Pi\left(\frac{t}{T}\right)$ with $T = 1$ s, is shown in blue. Note that this is the same rectangular time window as in Solution s.12-8(c)-(f), but here, it has been explicitly drawn. When applying $w_{\text{rect}}(t)$ to signal $x(t)$ according to (8.1) and sampling time-windowed signal $x_{w,\text{rect}}(t)$ with sampling frequency $f_s = 64$ Hz, one ends up with discrete-time sequence $x_{n,\text{rect}}$, shown by the burgundy dots (remember: $f_0 = 4.3$ Hz and $f_s = 64$ Hz, hence, aliasing is by far no issue here, as $f_s \gg f_0$). s.12-9



(b) In the right figure above, the Hann window (in blue), as a raised cosine window, with $w_{\text{rcos}}(t)$ as given by (8.5), has been applied to continuous-time signal $x(t)$ (not shown), resulting in the continuous burgundy curve, time-windowed signal $x_{w,\text{rcos}}(t)$. The burgundy dots represent discrete-time sequence $x_{n,\text{rcos}}$ obtained by sampling $x_{w,\text{rcos}}(t)$ (again remember that aliasing is by far no issue here).

(c) The two figures below show, for the two time-windowed (continuous-time) signals, the magnitude of the corresponding Fourier transform $|X_w(f)|$ (continuous curves), with $X_w(f) = X(f) * W(f)$, cf. (8.2), and $W_{\text{rect}}(f) = T\,\text{sinc}(Tf)$ for the rectangular window, at left, and $W_{\text{rcos}}(f)$ as given by (8.6) for the Hann window, at right. Note that the left figure is identical to the right figure in Solution S.12-8(c)-(f), as both figures show the Fourier transform result of the same time-windowed (and sampled) signal, i.e., the same rectangular window has been applied to the same signal (and the same sampling frequency f_s has been used). The stems represent the DFT result as magnitude spectrum $|X_k|$: they present $X_w(f)$ evaluated at N discrete frequencies in the interval $[-\frac{f_s}{2}, \frac{f_s}{2})$, assuming N even. The figures here show the spectrum only for frequencies of interest, i.e., $f \in [-6, 6]$ Hz. The frequency step of the DFT is $\Delta f = \frac{1}{T}$, in this case 1 Hz. The FFT result has been multiplied, in terms of amplitude, by Δt, (12.5), and divided by T, the latter to account for the window length, see Section 8.3, and thereby maintain equivalence with the Fourier transform of the continuous-time signal. In the right figure, the amplitude has in addition been multiplied by a factor of 2, as to account for the fact that the Hann window, compared to the rectangular window, causes a loss of in total half of the signal amplitude: for the coefficients of the window in discrete time we have: $\sum_{n=0}^{N} w_n = N$ for the rectangular window, and $\sum_{n=0}^{N} w_n = \frac{N}{2}$ for the Hann window. This can also be clearly seen upon comparison of the functions in blue in the figures in (a) and (b).



(d) Remember that an infinite-length, continuous-time cosine signal with frequency f_0 has a Fourier transform in-the-limit consisting of two Dirac pulses, one at frequency $f = -f_0$ and one at f_0, (5.16). From the figures above, it can be concluded that the Hann window is (indeed) effective in (largely) removing one aspect of the spectral leakage effect; the 'wobbles' in the spectrum are largely reduced. This is in line with the shapes of the Fourier transforms $W(f)$ shown in Figure 8.8. The Hann window has hardly any side lobes, though the price to pay is a main lobe which is wider than that of the rectangular window. This effect can also be clearly seen here in the figures above.

Basic Python code:

```
import numpy as np
from matplotlib import pyplot as plt

burgundy = [165/255,0/255,52/255]   # burgundy color

# demonstration using a simple cosine signal of leakage due to
```

```
# time-windowing (both in continuous time and discrete time)
# with a default rectangular (boxcar) window and a raised cosine
# time window (which does reduce (part of) the effect of leakage)

# x(t) = cos(2 pi f0 t)          this is an even function (unit amplitude)
# X(f) = 1/2 delta(f+f0) + 1/2 delta(f-f0)   so, Fourier transform is real

# for the first part of the basic Python code of this exercise,
# see the basic Python code in the solution of the previous exercise
# the code of this exercise is identical up until the line # zero-padding
# in the basic Python code in the solution of the previous exercise

# to be able to run the second part of the basic Python code of this
# exercise, some essential elements from the first part are included below
# without much explanation (explanation can be found in previous solution)

# constants
f0 = 4.3; T = 1; fs = 64; N = T*fs
# continuous-time cosine signal x(t)
t  = np.linspace(-2, 2, 401); xt = np.cos(2*np.pi*f0*t)
# continuous-time rectangular-windowed cosine signal x_w(t)
wt = np.heaviside(t+T/2,0)-np.heaviside(t-T/2,0); xwt = np.multiply(xt,wt)
# Fourier transform of continuous-time rectangular-windowed cosine X_w(f)
f  = np.linspace(-6, 6, 1201)
Xwf = T/2 * (np.sinc(T*(f+f0)) + np.sinc(T*(f-f0)))
# discrete-time (sampled) cosine signal x_n
t_vec = np.linspace(-T/2, T/2-1/fs, N); xn = np.cos(2*np.pi*f0*t_vec)
# DFT of sampled cosine signal x_n with amplitude scaling
f_vec = np.fft.fftfreq(N, 1/fs); Xk = np.fft.fft(xn) / N

# below follows the second part of the basic Python code of this exercise

# Hann window in continuous time and windowed (Hann) signal x_w(t)
Hwt  = ( np.multiply( ( np.heaviside(t+T/2, 0) - np.heaviside(t-T/2, 0) ),
                      ( 0.5 + 0.5 * np.cos(2*np.pi*t/T) ) ) )
xHwt = np.multiply(xt, Hwt)

# Fourier transform of continuous-time windowed (Hann) signal
# scaling by factor of 2 to account for Hann window
# (as overall Hann window eats away half of signal x(t) amplitude)
XHwf = 2 * 0.5 * ( T/2 * ( np.sinc(T*(f+f0)) + np.sinc(T*(f-f0)) )
                 + T/4 * ( np.sinc(T*(f+f0+1/T)) + np.sinc(T*(f-f0+1/T)) )
                 + T/4 * ( np.sinc(T*(f+f0-1/T)) + np.sinc(T*(f-f0-1/T)) ) )

# Hann window in discrete time and windowed (Hann) and sampled signal
Hwn  = ( np.multiply( ( np.heaviside(t_vec+T/2, 0)
                  - np.heaviside(t_vec-T/2, 0) ),
                      ( 0.5 + 0.5 * np.cos(2*np.pi*t_vec/T) ) ) )
xHwn = np.multiply(xn, Hwn)

# DFT of windowed (Hann) and sampled signal with amplitude scaling
# multiply Python's FFT result by Delta_t and divide by measurement time T
# to maintain equivalence with continuous-time signal x(t) (*Delta_t/T)
# and multiply by factor of 2 to account for Hann window
# (as overall Hann window eats away half of signal x(t) amplitude)
```

```
XHwk = 2 * np.fft.fft(xHwn) / N      # DFT; (*Delta_t/T equals divide by N)

plt.figure()
plt.subplot(3,2,3)       # rectangular window(ed signal) in time domain
plt.plot(t, xt, color=burgundy)
plt.plot(t, wt, color='b')
plt.plot(t_vec, xn, color=burgundy, marker='o', markersize=4)
plt.xlabel('$t$ [s]'); plt.ylabel('$w_{rect}(t),x(t),x_{n,rect}$')

plt.subplot(3,2,4)       # Hann window(ed signal) in time domain
plt.plot(t, xHwt, color=burgundy)
plt.plot(t, Hwt, color='b')
plt.plot(t_vec, xHwn, color=burgundy, marker='o', markersize=4)
plt.xlabel('$t$ [s]');plt.ylabel('$w_{rcos}(t),x_{w,rcos}(t),x_{n,rcos}$')

# in the plots below, Xk is overlaid directly on Xwf
# (which matches as fs >> f0; 'aliases' due to sampling are far away)

plt.subplot(3,2,5)       # (D)Fourier transform rectangular-windowed signal
plt.axvline( f0, color=burgundy, linewidth=0.5)
plt.axvline(-f0, color=burgundy, linewidth=0.5)
plt.plot(f, np.abs(Xwf), color=burgundy)
plt.stem(f_vec, np.abs(Xk), basefmt=' ', linefmt=burgundy, markerfmt=
    burgundy)
plt.xlabel('$f$ [Hz]'); plt.ylabel('$|X_{w,rect}(f)|,|X_{k,rect}|$')

plt.subplot(3,2,6)       # (D)Fourier transform Hann-windowed signal
plt.axvline( f0, color=burgundy, linewidth=0.5)
plt.axvline(-f0, color=burgundy, linewidth=0.5)
plt.plot(f, np.abs(XHwf), color=burgundy)
plt.stem(f_vec, np.abs(XHwk), basefmt=' ', linefmt=burgundy, markerfmt=
    burgundy)
plt.xlabel('$f$ [Hz]'); plt.ylabel('$|X_{w,rcos}(f)|,|X_{k,rcos}|$')
```

IV

Spectral estimation

13

Energy and power spectral density

Exercises

e| 13.2 Continuous-time signals

e.13-1

Consider the signal

$$x(t) = 3\cos\left(10\pi t + \frac{\pi}{6}\right) + 7\cos\left(16\pi t + \frac{\pi}{3}\right) + 4\cos\left(22\pi t + \frac{\pi}{4}\right).$$

(a) Determine and sketch the power spectral density $S(f)$ of this signal.

(b) Compute the power contained in the frequency range from 6 Hz to 14 Hz, *including* the 6 Hz and 14 Hz boundaries.

e.13-2

Consider signal $z(t)$ of Example 5.3 and the three signals of Exercise *e*.6-7 on page 96.

(a) Determine the energy spectral density $G(f)$ of each of these four signals.

(b) For signal $z(t)$, obtain the energy contained in the signal for frequencies $|f| < \frac{\alpha}{\pi}$. Assume $\alpha > 0$.
Hint: use the indefinite integral $\int \frac{1}{a^2 + b^2x^2}\,\mathrm{d}x = \frac{1}{ab}\arctan\left(\frac{b}{a}x\right)$.

(c) For signal $z(t)$, obtain the energy contained in the signal for frequencies $|f| < \frac{\alpha}{2\pi}$. Again assume $\alpha > 0$. Determine the percentage contribution of the energy contained in this frequency interval to the total energy E_z.

e| 13.c Computer exercises

e.13-3

Plot the energy spectral densities $G(f)$ of the four signals from Exercise *e*.13-2 and comment on their appearance.

Solutions

s| 13.2 Continuous-time signals

s.13-1

(a) Power spectral density function $S(f)$ for a periodic signal $x(t)$ is defined as: $S(f) = \sum_{k=-\infty}^{\infty} |X_k|^2 \delta(f - kf_0)$, (13.2). Remember that $|X_k| = |X_{-k}| = \frac{1}{2}A_k$, with $k \in \mathbb{N}^+$ and A_k as defined in $x(t) = a_0 + \sum_{k=1}^{\infty} A_k \cos(k\omega_0 t + \theta_k)$, (3.14), see (4.7) and pages 47 and 48 of the companion book, Theory. Hence, a non-zero phase θ_k does *not* change the amplitude and power spectra, or the power spectral density function for that matter.

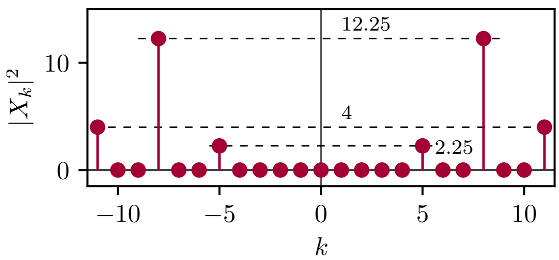
Signal $x(t) = 3\cos\left(10\pi t + \frac{\pi}{6}\right) + 7\cos\left(16\pi t + \frac{\pi}{3}\right) + 4\cos\left(22\pi t + \frac{\pi}{4}\right)$ has three components, with frequencies and amplitudes of $\{f_5 = 5$ Hz, $A_5 = 3\}$, $\{f_8 = 8$ Hz, $A_8 = 7\}$, and $\{f_{11} = 11$ Hz, $A_{11} = 4\}$. The signal has a double-sided amplitude line spectrum, with amplitude $\frac{A_k}{2}$ at frequency $-kf_0$ and amplitude $\frac{A_k}{2}$ at frequency kf_0, with $k = 5, 8, 11$ for the three individual components, and fundamental frequency $f_0 = 1$ Hz.

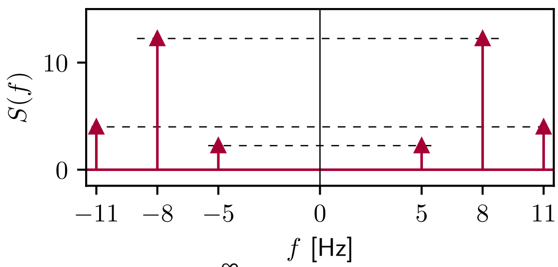
Below at left, $|X_k|^2$ is shown as a function of k, yielding the (discrete) power spectrum.

From the above, we know that $|X_k|^2 = \frac{A_k^2}{4}$. The power spectral density function $S(f)$ of $x(t)$ can then be written as:

$$
\begin{aligned}
S(f) &= \frac{A_5^2}{4}\big(\delta(f+f_5)+\delta(f-f_5)\big)+\frac{A_8^2}{4}\big(\delta(f+f_8)+\delta(f-f_8)\big)+\frac{A_{11}^2}{4}\big(\delta(f+f_{11})+\delta(f-f_{11})\big)\\
&= \frac{9}{4}\big(\delta(f+5)+\delta(f-5)\big)+\frac{49}{4}\big(\delta(f+8)+\delta(f-8)\big)+\frac{16}{4}\big(\delta(f+11)+\delta(f-11)\big)\\
&= 2.25\big(\delta(f+5)+\delta(f-5)\big)+12.25\big(\delta(f+8)+\delta(f-8)\big)+4\big(\delta(f+11)+\delta(f-11)\big)
\end{aligned}
$$

This power spectral density function $S(f)$ is shown below at right.





(b) The average power of a periodic signal $x(t)$ equals: $P_x = \sum_{k=-\infty}^{\infty} |X_k|^2$, (4.16). Hence, the power of signal $x(t)$ contained in the frequency range from 6 Hz to 14 Hz equals:

$$P_{6-14\text{Hz}} = \frac{7^2}{4} + \frac{4^2}{4} + \frac{4^2}{4} + \frac{7^2}{4} = 12.25 + 4 + 4 + 12.25 = 32.5$$

Similarly, the power between 0 Hz and 4 Hz equals: $P_{0-4\text{Hz}} = 0$

The power between 4 Hz and 6 Hz equals: $P_{4-6\text{Hz}} = \frac{3^2}{4} + \frac{3^2}{4} = 4.5$

s.13-2

(a) Energy spectral density $G(f)$ is defined as: $G(f) = |X(f)|^2$, (13.4). The Fourier transforms $X(f)$ of these signals were already determined in Example 5.3 for signal $z(t)$ and in Exercise e.6-7 (see Solution s.6-7) for the other three signals:

$$Z(f) = \frac{1}{\alpha + j2\pi f}, \quad A(f) = \frac{1}{\alpha - j2\pi f}, \quad B(f) = \frac{2\alpha}{\alpha^2 + (2\pi f)^2}, \quad \text{and} \quad C(f) = \frac{-j4\pi f}{\alpha^2 + (2\pi f)^2}$$

The magnitudes of these Fourier transforms were also already determined in Example 5.3 and in Exercise e.6-7 (see Solution s.6-7):

$$|Z(f)| = \frac{1}{\sqrt{\alpha^2 + (2\pi f)^2}} \qquad |A(f)| = \frac{1}{\sqrt{\alpha^2 + (2\pi f)^2}}$$
$$|B(f)| = \frac{2\alpha}{\alpha^2 + (2\pi f)^2} \qquad |C(f)| = \frac{4\pi|f|}{\alpha^2 + (2\pi f)^2}$$

From these, the energy spectral densities can easily be derived:

$$G_z(f) = |Z(f)|^2 = \left(\frac{1}{\sqrt{\alpha^2 + (2\pi f)^2}}\right)^2 = \frac{1}{\alpha^2 + (2\pi f)^2}$$
$$G_a(f) = |A(f)|^2 = \left(\frac{1}{\sqrt{\alpha^2 + (2\pi f)^2}}\right)^2 = \frac{1}{\alpha^2 + (2\pi f)^2}$$
$$G_b(f) = |B(f)|^2 = \left(\frac{2\alpha}{\alpha^2 + (2\pi f)^2}\right)^2 = \frac{(2\alpha)^2}{\left(\alpha^2 + (2\pi f)^2\right)^2}$$
$$G_c(f) = |C(f)|^2 = \left(\frac{4\pi|f|}{\alpha^2 + (2\pi f)^2}\right)^2 = \frac{(4\pi f)^2}{\left(\alpha^2 + (2\pi f)^2\right)^2}$$

(b) The energy contained in signal $z(t)$ for frequencies $|f| < \frac{\alpha}{\pi}$ can be obtained by integrating the energy spectral density from $-\frac{\alpha}{\pi}$ to $+\frac{\alpha}{\pi}$, (13.5):

$$E = \int_{-\frac{\alpha}{\pi}}^{\frac{\alpha}{\pi}} G_z(f)\,\mathrm{d}f = \int_{-\frac{\alpha}{\pi}}^{\frac{\alpha}{\pi}} \frac{1}{\alpha^2 + (2\pi f)^2}\,\mathrm{d}f = 2\int_0^{\frac{\alpha}{\pi}} \frac{1}{\alpha^2 + (2\pi f)^2}\,\mathrm{d}f$$

where in the latter step the evenness of the function was exploited.
Using the hint provided, with $a = \alpha$ and $b = 2\pi$, we obtain:

$$E = 2\frac{1}{\alpha 2\pi}\left[\arctan\left(\frac{2\pi}{\alpha}f\right)\right]_0^{\frac{\alpha}{\pi}} = \frac{1}{\alpha\pi}\left(\arctan(2) - \arctan(0)\right) = \frac{1}{\alpha\pi}\arctan(2)$$

(c) In a similar manner as in (b), but now for frequencies $|f| < \frac{\alpha}{2\pi}$:

$$E = 2\frac{1}{\alpha 2\pi}\left[\arctan\left(\frac{2\pi}{\alpha}f\right)\right]_0^{\frac{\alpha}{2\pi}} = \frac{1}{\alpha\pi}\left(\arctan(1) - \arctan(0)\right) = \frac{1}{\alpha\pi}\arctan(1) = \frac{1}{\alpha\pi}\frac{\pi}{4} = \frac{1}{4\alpha}$$

The total energy contained in the signal can be computed as well by taking the upper integral limit ∞:

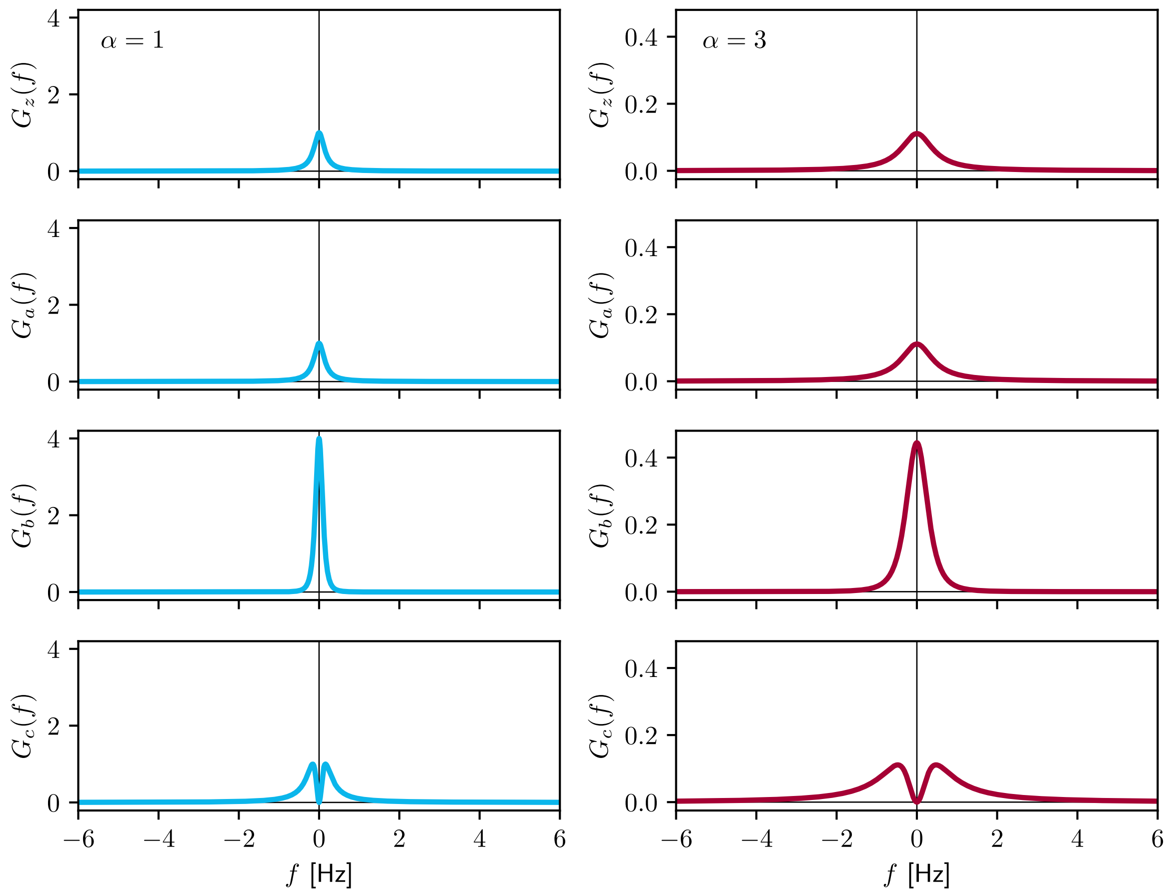
$$E_z = 2\frac{1}{\alpha 2\pi}\left[\arctan\left(\frac{2\pi}{\alpha}f\right)\right]_0^{\infty} = \frac{1}{\alpha\pi}\left(\arctan(\infty) - \arctan(0)\right) = \frac{1}{\alpha\pi}\arctan(\infty) = \frac{1}{\alpha\pi}\frac{\pi}{2} = \frac{1}{2\alpha}$$

The contribution of the energy contained in the frequency interval $|f| < \frac{\alpha}{2\pi}$ to the total energy E_z equals: $\left(1/(4\alpha)\right)/\left(1/(2\alpha)\right) = \frac{1}{2}$, which is a percentage contribution of 50%.
Similarly, the contribution of the energy contained in the frequency interval $|f| < \frac{\alpha}{\pi}$ (of (b)) to the total energy E_z equals:
$\left(\frac{1}{\alpha\pi}\arctan(2)\right)/\left(1/(2\alpha)\right) \approx 0.7048$, which is a percentage contribution of $\approx 70.5\%$.

s|13.c Computer exercises

s.13-3

The figure below shows the energy spectral densities $G(f)$ of the four signals for $\alpha = 1$ at left and $\alpha = 3$ at right. Note the difference in the vertical axis range for the left and right columns. One can see that the energy spectral densities are larger for $\alpha = 1$ than for $\alpha = 3$, as one could already have deduced from the expressions for $G(f)$ obtained in Exercise *e*.13-2(a) (see Solution *s*.13-2(a)) for the different signals. This also makes sense, since a larger time constant α means that the signal decays faster. The shapes of the energy spectral densities should not come as a surprise, since $G(f)$ equals $|Z(f)|^2$, $|A(f)|^2$, $|B(f)|^2$, $|C(f)|^2$, and $|Z(f)|$, $|A(f)|$, $|B(f)|$, $|C(f)|$ were already plotted in Exercise *e*.6-20 (see Solution *s*.6-20). Plotting the energy spectral density simply means plotting the squared amplitude spectrum of a signal. The energy spectral density is a real and even function of frequency. For a real-valued signal $x(t)$, we have: $|X(-f)| = |X(f)|$ and $G(f) = |X(f)|^2$, (13.4).



Basic Python code for signal $a(t)$:

```
import numpy as np
from matplotlib import pyplot as plt

burgundy = [165/255,0/255,52/255]  # burgundy color

# plot energy spectral density G_a(f) of exponential decay function a(t)

# Fourier transform A(f) of exponential decay function a(t)
alf1 = 1; alf3 = 3 # time constant alpha; larger constant --> faster decay
f = np.linspace(-10, 10, 2001) # frequency steps 0.01 Hz from -10 to 10 Hz
Af1 = 1 / (alf1 - 1j*2*np.pi*f); Af3 = 1 / (alf3 - 1j*2*np.pi*f)

# compute and plot energy spectral density G_a(f)
Ga1 = np.power(np.abs(Af1), 2); Ga3 = np.power(np.abs(Af3), 2)
plt.figure()
plt.subplot(3,2,1)
plt.plot(f, Ga1, color='b', linewidth=2)
plt.xlabel('$f$ [Hz]'); plt.ylabel('$G_a(f)$')
plt.subplot(3,2,2)
plt.plot(f, Ga3, color=burgundy, linewidth=2)
plt.xlabel('$f$ [Hz]'); plt.ylabel('$G_a(f)$')

# the remainder of the code for signals z(t), b(t), c(t) is highly similar
# (except for differences between the signals themselves) to the code
# above for a(t) and is therefore omitted to avoid excessive repetition
```

14

Spectral estimation

Exercises

e| 14.c Computer exercises

e.14-1

Consider the cosine signal

$$x(t) = \cos(2\pi f_0 t)$$

with a frequency of $f_0 = 5.2$ Hz, unit amplitude, and zero phase.

(a) Create discrete-time sequence x_n by sampling signal $x(t)$ with a sampling frequency of $f_s = 40$ Hz for a duration of $T = 100$ s.

(b) Add zero-mean noise to the $N = f_s T$ samples of the discrete-time cosine signal to simulate a signal that might be measured in practice. Do this for two different noise levels: one with standard deviation $\sigma = 1$ and another with $\sigma = 3$.
Hint: in Python, one can use the function `numpy.random.normal` to generate noise from a normal distribution, i.e., create Additive White Gaussian Noise (AWGN), see Appendix L.

(c) Plot the two discrete-time signals as a function of the 'restored' continuous time t in [s] on the horizontal axis. Zoom in on the first 3 seconds to ensure that the plot is not too dense.

(d) Compute and plot the basic periodogram (expressed in decibel) for both signals.
Hint: in Python, use function `scipy.signal.periodogram` with the default option `window='boxcar'`, and specifically `detrend = False` and `return_onesided = False` as, by default, we consider two-sided spectra (though we plot only the part for positive frequencies).

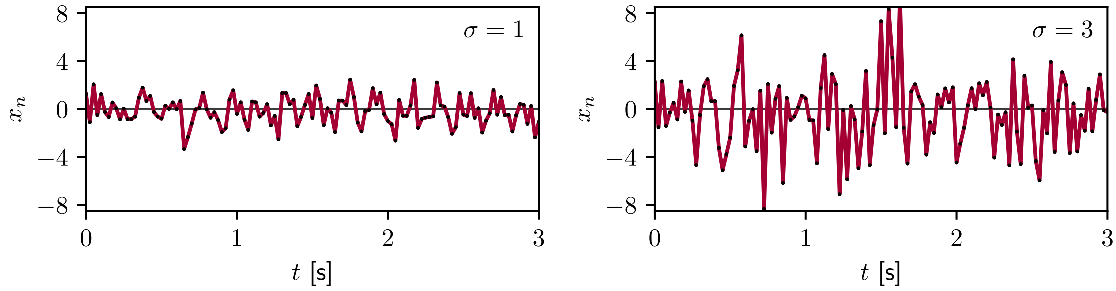
(e) Compute and plot the Welch periodogram (again expressed in decibel) for both signals, using $M = 10$ segments with no overlap.
Hint: in Python, use function `scipy.signal.Welch`, with in addition to the periodogram options in (d), the options `nperseg=N//M` and `noverlap=0`.

Solutions

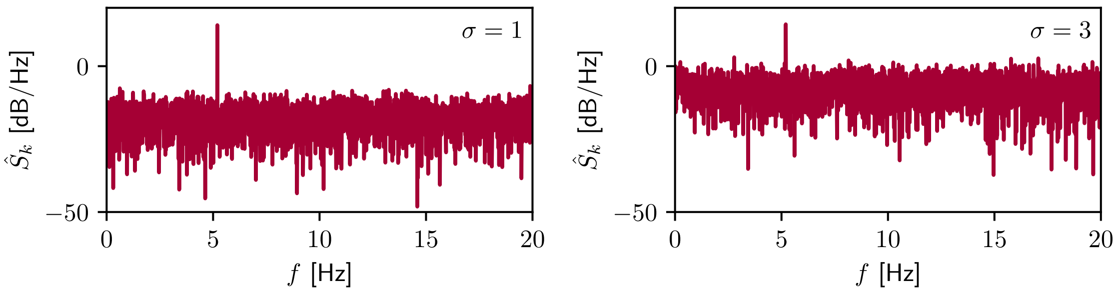
s| 14.c Computer exercises

s.14-1

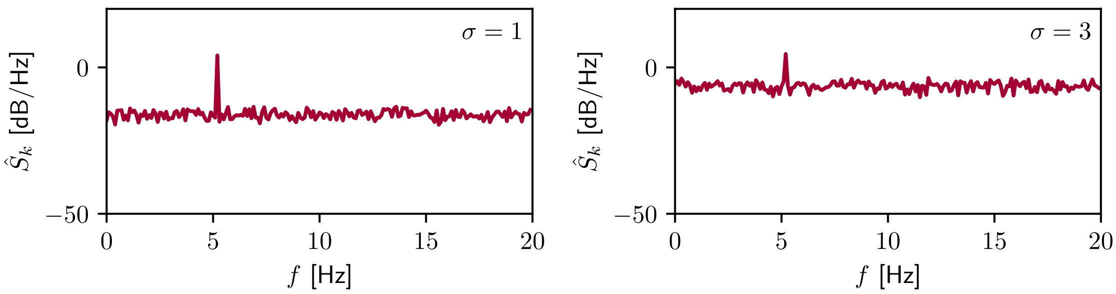
(a)-(c) The figure below shows the two discrete-time signals with added noise ($\sigma = 1$ at left and $\sigma = 3$ at right). Note that the discrete-time samples (black dots) are connected here by line segments for clearer visualization (it is *not* a continuous-time signal). In both plots we see a considerable amount of noise. In the right plot for the case with $\sigma = 3$ we can hardly recognize the cosine signal.



(d) Both basic periodograms below (for the case of $\sigma = 1$ at left and $\sigma = 3$ at right) show a so-called flat spectrum across the entire frequency range, as expected with white noise input. Clearly, the noise level is larger for the case of $\sigma = 3$ than for $\sigma = 1$. In both cases there is a clear peak, well above the noise, at the frequency $f_0 = 5.2$ Hz of the cosine.



(e) The figure below shows the Welch periodograms for the two discrete-time signals with added noise ($\sigma = 1$ at left and $\sigma = 3$ at right). Dividing the data record into segments and averaging the periodograms is clearly beneficial in reducing the noise in the spectral estimate. Note that the frequency of cosine signal $x(t)$ was set to $f_0 = 5.2$ Hz, and that with $M = 10$ segments, the segment duration is $\frac{T}{M} = \frac{100}{10} = 10$ s. Hence, within a single segment the frequency step is $\Delta f = \frac{M}{T} = 0.1$ Hz, and we (still) see a sharp peak at $f_0 = 5.2$ Hz, i.e., also with a single segment the frequency resolution is sufficient (and with these results we do not notice any leakage).



Basic Python code:

```
import numpy as np
from scipy import signal
from matplotlib import pyplot as plt

burgundy = [165/255,0/255,52/255]                  # burgundy color

# discrete-time sampled cosine signal x_n
f0 = 5.2                                           # cosine frequency [Hz]
fs = 40                                            # sampling frequency [Hz]
T  = 100                                           # signal duration T [s]
N  = T * fs                                        # number of samples
t  = np.arange(0, T, 1/fs)                         # sample time vector [s]
xn = np.cos(2*np.pi*f0*t)                          # x(t) = cos(2 pi f0 t)

# discrete-time sampled signal with Additive White Gaussian Noise
# AWGN in discrete time, dt=1/fs, we actually use band-limited white noise
# S_noise(f) = N0/2 * \Pi(f/fs), pulse with width fs, reaching to +/-fs/2
# on either side, and
# R(\tau) = N0/2 * fs \sinc(fs \tau) = sigma**2 \sinc(fs \tau), hence with
# a 'null' at \tau = 1/fs = \delta t; we sample at the nulls of the sinc
xn1t = xn + np.random.normal(0, 1.0, N) # cosine+noise (zero mean,sigma=1)
xn3t = xn + np.random.normal(0, 3.0, N) #                         sigma=3

# basic periodogram
fn1, Sn1f = signal.periodogram(xn1t, fs, window='boxcar', return_onesided
   = False, detrend = False)
fn3, Sn3f = signal.periodogram(xn3t, fs, window='boxcar', return_onesided
   = False, detrend = False)

Sn1f_dB = 10*np.log10(Sn1f); Sn3f_dB = 10*np.log10(Sn3f)      # in decibel

# Welch periodogram
M = 10          # number of (non-overlapping) segments for Welch periodogram
fn1_w, Sn1f_w = signal.welch(xn1t, fs, window='boxcar', nperseg=N//M,
   noverlap=0, return_onesided = False, detrend = False)
fn3_w, Sn3f_w = signal.welch(xn3t, fs, window='boxcar', nperseg=N//M,
   noverlap=0, return_onesided = False, detrend = False)

Sn1f_w_dB =10*np.log10(Sn1f_w); Sn3f_w_dB =10*np.log10(Sn3f_w)# in decibel

# plot signal as a function of time (actually discrete time)
plt.figure()
plt.subplot(3,2,1)
plt.plot(t, xn1t, color=burgundy)
plt.plot(t, xn1t, '.', color='k', markersize=1)
plt.xlim([0, 3])    # show only first 3 seconds (otherwise graph too dense)
plt.xlabel('$t$ [s]'); plt.ylabel('$x_n$')

plt.subplot(3,2,2)
plt.plot(t, xn3t, color=burgundy)
plt.plot(t, xn3t, '.', color='k', markersize=1)
plt.xlim([0, 3])    # show only first 3 seconds (otherwise graph too dense)
plt.xlabel('$t$ [s]'); plt.ylabel('$x_n$')
```

```python
# plot (basic) periodogram (show only for positive frequencies)
plt.subplot(3,2,3)
plt.plot(fn1[fn1>=0], Sn1f_dB[fn1>=0], color=burgundy)
plt.xlabel('$f$ [Hz]'); plt.ylabel('$\hat{S}_k$ [dB/Hz]')

plt.subplot(3,2,4)
plt.plot(fn3[fn3>=0], Sn3f_dB[fn3>=0], color=burgundy)
plt.xlabel('$f$ [Hz]'); plt.ylabel('$\hat{S}_k$ [dB/Hz]')

# plot Welch periodogram (show only for positive frequencies)
plt.subplot(3,2,5)
plt.plot(fn1_w[fn1_w>=0], Sn1f_w_dB[fn1_w>=0], color=burgundy)
plt.xlabel('$f$ [Hz]'); plt.ylabel('$\hat{S}_k$ [dB/Hz]')

plt.subplot(3,2,6)
plt.plot(fn3_w[fn3_w>=0], Sn3f_w_dB[fn3_w>=0], color=burgundy)
plt.xlabel('$f$ [Hz]'); plt.ylabel('$\hat{S}_k$ [dB/Hz]')
```

15

Spectrogram

Exercises

e| 15.c Computer exercises

e.15-1

A spectrogram can be used to obtain an impression about the stationarity or non-stationarity of a signal. The spectrogram offers a visual inspection of the signal in both the time domain and the frequency domain.

In this exercise, we use a cosine signal, with unit amplitude $A = 1$ and zero phase $\theta = 0$, but one of which the frequency changes with time, in this case linearly, according to $f(t) = f_0 + rt$, referred to as a frequency sweep, with start frequency f_0, at time $t = 0$ s, and frequency rate of change $r = \frac{f_1 - f_0}{T}$, with stop frequency f_1, at time $t = T$, and sweep duration T.

This is an example of a so-called chirp signal. Chirp signals are often used in radar, and by bats for echolocation to navigate and hunt.

(a) Create the above described chirp signal as a (discrete) function of time x_n, with sampling frequency $f_s = 100$ Hz, start frequency $f_0 = 5$ Hz, stop frequency $f_1 = 25$ Hz and sweep duration $T = 10$ s, for the duration of just one sweep.
Hint: in Python, one can use the function `scipy.signal.chirp` to generate a frequency-swept cosine.

(b) Plot discrete-time chirp signal x_n as a function of the 'restored' continuous time t in [s] on the horizontal axis.

The phase of a harmonic follows by integration of its frequency

$$x(t) = \cos\Big(2\pi \int_0^t f(\tau)\,\mathrm{d}\tau\Big) = \cos\Big(2\pi \int_0^t (f_0 + r\tau)\,\mathrm{d}\tau\Big) = \cos\Big(2\pi\big(f_0 t + \frac{r}{2}t^2\big)\Big)$$

and is a quadratic function of time. When the frequency rate equals zero, $r = 0$, we are back at the usual plain cosine $x(t) = \cos(2\pi f_0 t)$, with constant frequency f_0.
The chirp signal obviously is non-stationary, as its frequency changes with time.

(c) Create and plot the spectrogram of the chirp signal, once with a segment duration of 0.4 s, and once with a segment duration of 1.0 s, both using a rectangular window and without overlap.

Hint: in Python, it is most convenient to use the function `specgram` from the `matplotlib.pyplot` library, as it automatically plots the spectrogram after computation, unlike the function `spectrogram` from `scipy.signal.ShortTimeFFT`, which computes the spectrogram but does not plot it. For example: `specgram(x, NFFT=window_length, Fs=fs, window=signal.get_window('boxcar',window_length), noverlap=0)`. Here, we also make use of the function `signal.get_window` from the `scipy` library. The output of `specgram` is, by default, a plot (heat map) of a single-sided Power Spectral Density (PSD) estimate on a logarithmic scale, with unit [dB/Hz].

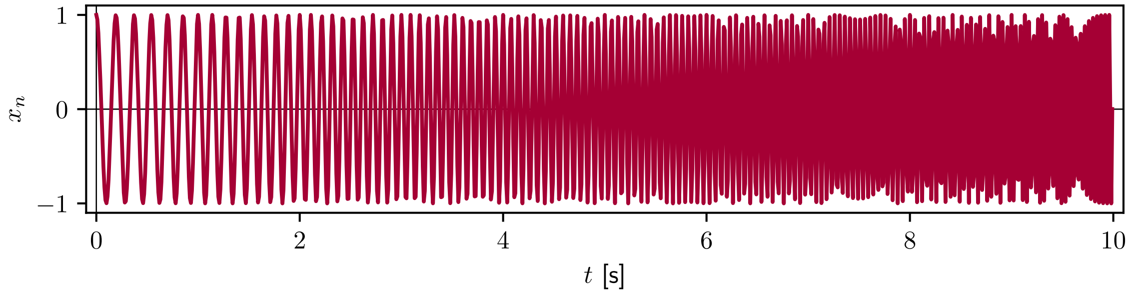
(d) Comment on the temporal and spectral resolution (time versus frequency) and differences between these two spectrograms.

Solutions

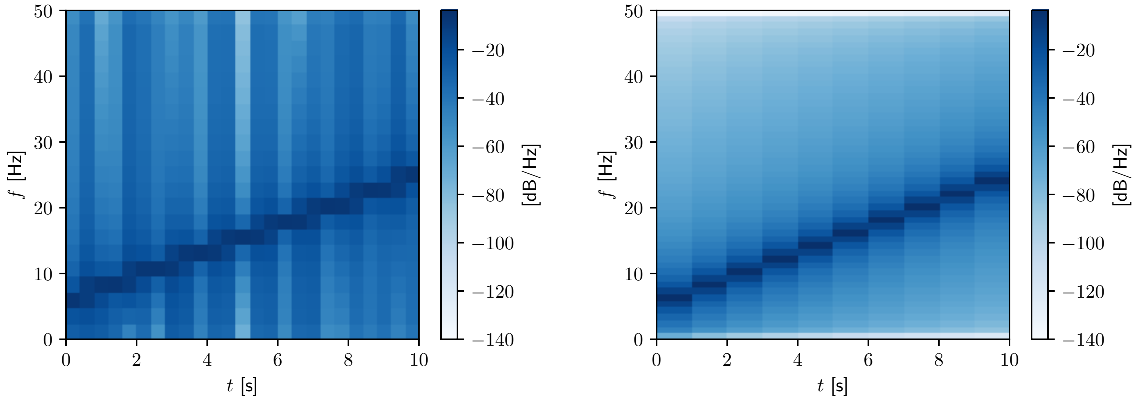
s|15.c Computer exercises

S.15-1

(a)-(b) The figure below shows discrete-time chirp signal x_n. It appears to be a continuous-time signal, as the samples are plotted as a single continuous curve for clearer visualization, but it *is* a discrete-time signal, with $f_s = 100$ Hz.



(c) Below, the spectrogram of the chirp signal x_n obtained using a segment duration of 0.4 s is shown at left, and the spectrogram obtained using a segment duration of 1.0 s is shown at right. The *single-sided* periodograms are shown in the vertical direction, stacked next to each other along the time axis. The value of the PSD is indicated by the shade of blue (see the color bars to the right of the respective spectrograms).



(d) Both spectrograms show roughly the same behavior, by means of the dark blue band, namely, a frequency which is changing linearly in time. With a segment duration of 0.4 s, the time axis is divided into 25 bins, and the frequency step size is $\Delta f = \frac{1}{T_M}$ Hz, where T_M is the segment duration, hence $\Delta f = 2.5$ Hz. With a segment duration of 1.0 s, the time axis is divided into only 10 bins (poorer time resolution), and the frequency step size is $\Delta f = 1.0$ Hz (better frequency resolution). The price to pay for improved frequency resolution is reduced temporal resolution (longer segments).

Basic Python code:

```
import numpy as np
from matplotlib import pyplot as plt
from scipy import signal
from scipy.signal import chirp

burgundy = [165/255,0/255,52/255]   # burgundy color
```

```
# chirp signal parameters
fs = 100                # sampling frequency [Hz]
T  = 10                 # measurement/sweep duration [s]
f0 = 5                  # start frequency [Hz]
f1 = 25                 # end frequency [Hz]

# chirp signal as (discrete) function of time
# frequency of this signal is a linear function of time
t = np.linspace(0, T, T*fs, endpoint=False) # last time point not included
x = chirp(t, f0=f0, f1=f1, t1=T, method='linear')   # linear chirp signal

# plot of chirp signal as (discrete) function of time
plt.figure()
plt.subplot(3,1,1)
plt.plot(t, x, color=burgundy, linewidth=1.5)
plt.xlabel('$t$ [s]'); plt.ylabel('$x_n$')

# window lengths specified in number of samples
segment_duration1 = 0.4; segment_duration2 = 1.0    # segment duration [s]
window_length04 = int(segment_duration1*fs)         # 0.4 [s]
window_length10 = int(segment_duration2*fs)         # 1.0 [s]

# spectrogram obtained using rectangular (boxcar) window of length 0.4 s
# no zero-padding, no overlap between adjacent segments of data
# PSD in dB, single-sided
plt.figure()
plt.specgram(x, NFFT=window_length04, Fs=fs, window=signal.get_window('
   boxcar', window_length04), noverlap=0, cmap='Blues')
plt.xlabel('$t$ [s]'); plt.ylabel('$f$ [Hz]')
cbar = plt.colorbar(); cbar.set_label('[dB/Hz]')

# spectrogram obtained using rectangular (boxcar) window of length 1.0 s
plt.figure()
plt.specgram(x, NFFT=window_length10, Fs=fs, window=signal.get_window('
   boxcar', window_length10), noverlap=0, cmap='Blues')
plt.xlabel('$t$ [s]'); plt.ylabel('$f$ [Hz]')
cbar = plt.colorbar(); cbar.set_label('[dB/Hz]')
```

V

Linear systems

16

Linear time-invariant systems

Exercises

e|16.1 System properties

e.16-1

Determine for each of the systems described below:

(1) What its order is;

(2) Whether it is time-invariant (fixed) or time-varying;

(3) Whether it is linear or nonlinear.

(a) $4\frac{\mathrm{d}^2 y(t)}{\mathrm{d}t^2} + 5y(t) = 2\frac{\mathrm{d}^3 x(t)}{\mathrm{d}t^3} + 3x(t)$

(b) $2y(t) + 9 = 8\frac{\mathrm{d}x(t)}{\mathrm{d}t} + x(t)$

(c) $7y(t) + 3\int_{-\infty}^{t} y(\tau)\,\mathrm{d}\tau = 4x(t)$

(d) $2\frac{\mathrm{d}^3 y(t)}{\mathrm{d}t^3} + y(t)\frac{\mathrm{d}y(t)}{\mathrm{d}t} + y(t) = 2x(t)$

(e) $\frac{3}{2}\frac{\mathrm{d}y(t)}{\mathrm{d}t} + t^3 y(t) = \frac{3}{2}\int_{-\infty}^{t} x(\tau)\,\mathrm{d}\tau$

e.16-2

A system is described by the input-output relation

$$y(t) = \tfrac{1}{3}x(t^{\frac{1}{3}})$$

Is this system causal or non-causal? Justify your answer.

e.16-3

A system is described by the input-output relation

$$y(t) = 2x\left(t + \tfrac{1}{2}\right) + 4$$

Is this system... (a) ... linear? (b) ... causal?

Justify your answers.

e.16-4

A system is described by the input-output relation

$$y(t) = 3x(t^3)$$

Is this system... (a) ... time-invariant? (b) ... linear? (c) ... causal?

Justify your answers.

e.16-5 Consider the six systems described below. For each of these systems, determine whether each of the following properties is true or false: (1) dynamic, (2) time-invariant, (3) linear, and (4) causal. If true, check the appropriate box in the table at the bottom of this exercise. Also write the order of the system in the bottom row.

(a) $3\frac{\mathrm{d}^2y(t)}{\mathrm{d}t^2} + 2\frac{\mathrm{d}y(t)}{\mathrm{d}t} + y(t) + 4\int_{-\infty}^{t} y(\tau)\,\mathrm{d}\tau = 2x(t)$

(b) $5\frac{\mathrm{d}^3y(t)}{\mathrm{d}t^3} + 3\frac{\mathrm{d}^2y(t)}{\mathrm{d}t^2} + 4\frac{\mathrm{d}y(t)}{\mathrm{d}t} + 2y^2(t) = 3x(t)$

(c) $4\frac{\mathrm{d}^2y(t)}{\mathrm{d}t^2} + 2t\frac{\mathrm{d}y(t)}{\mathrm{d}t} + 3y(t) = 5x(t)$

(d) $4y(t)\frac{\mathrm{d}^2y(t)}{\mathrm{d}t^2} + 2t\frac{\mathrm{d}y(t)}{\mathrm{d}t} + 3y(t) = 5x(t)$

(e) $2y(t) = x(t^2) + 2x(t)$

(f) $4y(t)\frac{\mathrm{d}^2y(t)}{\mathrm{d}t^2} + 2t\frac{\mathrm{d}y(t)}{\mathrm{d}t} + 3y(t) = 5x(2t)$

	System					
Property	(a)	(b)	(c)	(d)	(e)	(f)
(1) Dynamic						
(2) Time-invariant						
(3) Linear						
(4) Causal						
(5) Order						

e.16-6 Consider the six systems described below. For each of these systems, determine whether each of the following properties is true or false: (1) dynamic, (2) time-invariant, (3) linear, and (4) causal. Make a table similar to the one in Exercise e.16-5, and if a property is true, check the appropriate box in the table. Also write the order of the system in the bottom row.

(a) $y(t) = 3x(t-3)$

(b) $y(t) = \sqrt{x(t)} + \frac{1}{2}$

(c) $\frac{\mathrm{d}y(t)}{\mathrm{d}t} + \sqrt{t}y(t) = x(t)$

(d) $\frac{\mathrm{d}^2y(t)}{\mathrm{d}t^2} + \frac{1}{t}y(t) = x(t+1)$

(e) $y(t) = \int_{-\infty}^{t-1} x(\tau)\,\mathrm{d}\tau$

(f) $\frac{\mathrm{d}^3y(t)}{\mathrm{d}t^3} + y(t)\frac{\mathrm{d}y(t)}{\mathrm{d}t} + 2y(t) + \int_{-\infty}^{t} y(\tau)\,\mathrm{d}\tau = 4x(t)$

e.16-7 A system is described by the input-output relation

$$y(t) = 2x(t)\cos(25\pi t)$$

Is this system...

(a) ... dynamic? (b) ... time-invariant? (c) ... linear? (d) ... causal?

Justify your answers.

e| 16.2 System response

e.16-8

Using the properties of linearity and time invariance, show that the output $y(t)$ of a linear time-invariant system can be obtained as the convolution of input $x(t)$ with impulse response function $h(t)$, i.e.:

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,\mathrm{d}\tau \qquad (16.6)$$

Hint: start from expressing input $x(t)$ as the convolution of $x(t)$ with the Dirac delta function $\delta(t)$, i.e.:

$$x(t) = x(t) * \delta(t) = \int_{-\infty}^{\infty} x(\lambda)\delta(t-\lambda)\,\mathrm{d}\lambda$$

e| 16.c Computer exercises

e.16-9

Given is a cosine signal

$$x(t) = \cos(2\pi f_0 t)$$

as input to the LTI system of Example 16.5, with impulse response function

$$h(t) = \alpha e^{-\alpha t}u(t) \qquad \text{for} \quad \alpha > 0, \qquad \text{cf. (16.11)}.$$

Compute and visualize the output signal $y(t)$ for $f_0 = 1$ Hz and $\alpha = 1$.

e.16-10

Given is the sum of two cosine signals

$$x(t) = \cos(2\pi f_1 t) + \cos(2\pi f_2 t)\,,$$

both with unit amplitude and zero initial phase, as input to the LTI system of Example 16.5, with impulse response function

$$h(t) = \alpha e^{-\alpha t}u(t) \qquad \text{for} \quad \alpha > 0, \qquad \text{cf. (16.11)}.$$

Compute, visualize and analyze the output signal $y(t)$ for $f_1 = 1$ Hz, $f_2 = \frac{1}{2}$ Hz and $\alpha = 1$.

Solutions

S | 16.1 System properties

S.16-1

(a) (1) The order is 2 (no integrals, $\frac{d^2y(t)}{dt^2}$ highest derivative of output $y(t)$)
(2) Time-invariant (DE coefficients are all constants)
(3) Linear:
Response $y_1(t)$ to $x_1(t)$ satisfies: $4\frac{d^2y_1(t)}{dt^2} + 5y_1(t) = 2\frac{d^3x_1(t)}{dt^3} + 3x_1(t)$

Response $y_2(t)$ to $x_2(t)$ satisfies: $4\frac{d^2y_2(t)}{dt^2} + 5y_2(t) = 2\frac{d^3x_2(t)}{dt^3} + 3x_2(t)$

Multiplying by arbitrary constants α_1 and $\alpha_2 \in \mathbb{R}$ and then adding them together (omitting (t) for brevity) yields:

$$\alpha_1 4\frac{d^2y_1}{dt^2} + \alpha_2 4\frac{d^2y_2}{dt^2} + \alpha_1 5y_1 + \alpha_2 5y_2 = \alpha_1 2\frac{d^3x_1}{dt^3} + \alpha_2 2\frac{d^3x_2}{dt^3} + \alpha_1 3x_1 + \alpha_2 3x_2$$
$$\Rightarrow \quad 4\frac{d^2}{dt^2}(\alpha_1y_1 + \alpha_2y_2) + 5(\alpha_1y_1 + \alpha_2y_2) = 2\frac{d^3}{dt^3}(\alpha_1x_1 + \alpha_2x_2) + 3(\alpha_1x_1 + \alpha_2x_2)$$

Hence, the response to the input $\alpha_1x_1(t) + \alpha_2x_2(t)$ equals $\alpha_1y_1(t) + \alpha_2y_2(t)$. The superposition principle, (16.5), holds, the system is linear.

(b) (1) The order is zero (no integrals, no derivative of output $y(t)$)
(2) Time-invariant (DE coefficients are all constants)
(3) Nonlinear:
Response $y_1(t)$ to $x_1(t)$ satisfies: $2y_1(t) + 9 = 8\frac{dx_1(t)}{dt} + x_1(t)$

Response $y_2(t)$ to $x_2(t)$ satisfies: $2y_2(t) + 9 = 8\frac{dx_2(t)}{dt} + x_2(t)$

Multiplying by arbitrary constants α_1 and $\alpha_2 \in \mathbb{R}$ and then adding them together (omitting (t) for brevity) yields:

$$\alpha_1 2y_1 + \alpha_2 2y_2 + \alpha_1 9 + \alpha_2 9 = \alpha_1 8\frac{dx_1}{dt} + \alpha_2 8\frac{dx_2}{dt} + \alpha_1x_1 + \alpha_2x_2$$
$$\Rightarrow \quad 2(\alpha_1y_1 + \alpha_2y_2) + 9(\alpha_1 + \alpha_2) = 8\frac{d}{dt}(\alpha_1x_1 + \alpha_2x_2) + (\alpha_1x_1 + \alpha_2x_2)$$

We cannot put the above equation into the same form as the original DE for arbitrary values of α_1 and α_2, it can only be done when $(\alpha_1 + \alpha_2) = 1$. The superposition principle, (16.5), does *not* hold. Hence, the system is nonlinear, caused by the term $+9$ on the left-hand side of the original DE.

(c) (1) The order is 1 (take derivative: $7\frac{dy(t)}{dt} + 3y(t) = 4\frac{dx(t)}{dt}$)
(2) Time-invariant (DE coefficients are all constants)
(3) Linear:
After having taken the derivative:
Response $y_1(t)$ to $x_1(t)$ satisfies: $7\frac{dy_1(t)}{dt} + 3y_1(t) = 4\frac{dx_1(t)}{dt}$

Response $y_2(t)$ to $x_2(t)$ satisfies: $7\frac{dy_2(t)}{dt} + 3y_2(t) = 4\frac{dx_2(t)}{dt}$

Multiplying by arbitrary constants α_1 and $\alpha_2 \in \mathbb{R}$ and then adding them together (omitting (t) for brevity) yields:

$$\alpha_1 7\frac{dy_1}{dt} + \alpha_2 7\frac{dy_2}{dt} + \alpha_1 3y_1 + \alpha_2 3y_2 = \alpha_1 4\frac{dx_1}{dt} + \alpha_2 4\frac{dx_2}{dt}$$
$$\Rightarrow \quad 7\frac{d}{dt}(\alpha_1y_1 + \alpha_2y_2) + 3(\alpha_1y_1 + \alpha_2y_2) = 4\frac{d}{dt}(\alpha_1x_1 + \alpha_2x_2)$$

Hence, the response to the input $\alpha_1x_1(t) + \alpha_2x_2(t)$ equals $\alpha_1y_1(t) + \alpha_2y_2(t)$. The superposition principle, (16.5), holds, the system is linear.

(d) (1) The order is 3 (no integrals, $\frac{\mathrm{d}^3y(t)}{\mathrm{d}t^3}$ highest derivative of output $y(t)$)

(2) Time-invariant (DE coefficients are all constants)

(One might initially think that, due to the fact that the 'coefficient' multiplying $\frac{\mathrm{d}y(t)}{\mathrm{d}t}$ in the second term equals $y(t)$, which depends on time, that the system is time-varying. This is not the case, however, as output $y(t)$ is, for a given input signal $x(t)$ and when assuming that the system is at rest before the input is active, always the same. That is, although $y(t)$ obviously changes in time when an input signal is put on the system, apart from the input signal it does not depend on anything else.)

(3) Nonlinear:

Response $y_1(t)$ to $x_1(t)$ satisfies: $2\frac{\mathrm{d}^3y_1(t)}{\mathrm{d}t^3} + y_1(t)\frac{\mathrm{d}y_1(t)}{\mathrm{d}t} + y_1(t) = 2x_1(t)$

Response $y_2(t)$ to $x_2(t)$ satisfies: $2\frac{\mathrm{d}^3y_2(t)}{\mathrm{d}t^3} + y_2(t)\frac{\mathrm{d}y_2(t)}{\mathrm{d}t} + y_2(t) = 2x_2(t)$

Multiplying by arbitrary constants α_1 and $\alpha_2 \in \mathbb{R}$ and then adding them together (omitting (t) for brevity) yields:

$$\alpha_1 2\frac{\mathrm{d}^3y_1}{\mathrm{d}t^3} + \alpha_2 2\frac{\mathrm{d}^3y_2}{\mathrm{d}t^3} + \alpha_1 y_1\frac{\mathrm{d}y_1}{\mathrm{d}t} + \alpha_2 y_2\frac{\mathrm{d}y_2}{\mathrm{d}t} + \alpha_1 y_1 + \alpha_2 y_2 = \alpha_1 2x_1 + \alpha_2 2x_2$$

$$\Rightarrow\quad 2\frac{\mathrm{d}^3}{\mathrm{d}t^3}\left(\alpha_1 y_1 + \alpha_2 y_2\right) + \alpha_1 y_1\frac{\mathrm{d}y_1}{\mathrm{d}t} + \alpha_2 y_2\frac{\mathrm{d}y_2}{\mathrm{d}t} + \left(\alpha_1 y_1 + \alpha_2 y_2\right) = 2\left(\alpha_1 x_1 + \alpha_2 x_2\right)$$

We cannot put the above equation into the same form as the original DE for arbitrary values of α_1 and α_2 (in fact, from here the original DE only appears when either α_1 or α_2 is zero). The superposition principle, (16.5), does *not* hold, the system is nonlinear.

(e) (1) The order is 2 (take derivative, because of integral on input $x(t)$)

(2) Time-varying (coefficient multiplied by $y(t)$, i.e., t^3, clearly depends on time, hence the system dynamics change in time)

(3) Linear:

Response $y_1(t)$ to $x_1(t)$ satisfies: $\frac{3}{2}\frac{\mathrm{d}y_1(t)}{\mathrm{d}t} + t^3y_1(t) = \frac{3}{2}\int_{-\infty}^{t} x_1(\tau)\,\mathrm{d}\tau$

Response $y_2(t)$ to $x_2(t)$ satisfies: $\frac{3}{2}\frac{\mathrm{d}y_2(t)}{\mathrm{d}t} + t^3y_2(t) = \frac{3}{2}\int_{-\infty}^{t} x_2(\tau)\,\mathrm{d}\tau$

Multiplying by arbitrary constants α_1 and $\alpha_2 \in \mathbb{R}$ and adding them together yields:

$$\alpha_1\frac{3}{2}\frac{\mathrm{d}y_1(t)}{\mathrm{d}t} + \alpha_2\frac{3}{2}\frac{\mathrm{d}y_2(t)}{\mathrm{d}t} + \alpha_1 t^3y_1(t) + \alpha_2 t^3y_2(t) = \alpha_1\frac{3}{2}\int_{-\infty}^{t} x_1(\tau)\,\mathrm{d}\tau + \alpha_2\frac{3}{2}\int_{-\infty}^{t} x_2(\tau)\,\mathrm{d}\tau$$

$$\Rightarrow\quad \frac{3}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\alpha_1 y_1(t) + \alpha_2 y_2(t)\big) + t^3\big(\alpha_1 y_1(t) + \alpha_2 y_2(t)\big) = \frac{3}{2}\int_{-\infty}^{t}\big(\alpha_1 x_1(\tau) + \alpha_2 x_2(\tau)\big)\,\mathrm{d}\tau$$

Hence, the response to the input $\alpha_1 x_1(t) + \alpha_2 x_2(t)$ equals $\alpha_1 y_1(t) + \alpha_2 y_2(t)$. The superposition principle, (16.5), holds, the system is linear.

S.16-2

This system is non-causal. This follows from looking at the system equation for $0 < t < 1$ s. Take for instance $t = \frac{1}{3}$ s: $y\left(\frac{1}{3}\right) = \frac{1}{3}x\left(\left(\frac{1}{3}\right)^{\frac{1}{3}}\right) = \frac{1}{3}x\left(\frac{1}{\sqrt[3]{3}}\right)$. We see that the system output at $t = \frac{1}{3}$ s depends on the system input at $t = \frac{1}{\sqrt[3]{3}} \approx 0.693$ s. In other words: the system output at this time instant depends on a *future* value of the input signal.

$\mathcal{S}$.16-3

(a) This system is nonlinear, caused by the term $+4$ on the right-hand side (see also Solution $\mathcal{S}$.16-1(b)).

(b) This system is non-causal, as the system output $y(t)$ at a particular time depends on a *future* value of the input signal. Take for instance $t = 0$ s: $y(0) = 2x\big(0 + \frac{1}{2}\big) + 4$. We see that the system output at $t = 0$ s depends on the system input at $t = \frac{1}{2}$ s, a value that lies in the future.

$\mathcal{S}$.16-4

(a) This system is time-varying:
A system is time-invariant if its input-output relation does not change with time. The output of the system to an input at $t = t$ should then be equal to the output of the system to that same input but at another time $t-\tau$ (and the output being equally shifted in time), cf. (16.3). Note that in the discussion of whether a system is time-invariant or not, the system is considered to be 'at rest' before the input is applied.

Assume that the system is 'at rest' before input signal $x_1(t)$ is applied to the system. The response is then $y_1(t) = 3x_1(t^3)$. When considering a delayed (time-shifted) output, we get: $y_1(t-\tau) = 3x_1\big((t-\tau)^3\big)$. Now, again assume that the system is 'at rest' and we look at the response to a delayed input signal $x_2(t) = x_1(t-\tau)$, we get: $y_2(t) = 3x_2(t^3) = 3x_1(t^3-\tau)$. Clearly, $y_1(t-\tau) \neq y_2(t)$, they are not the same: the system is therefore time-varying.

(b) This system is linear:

Response $y_1(t)$ to $x_1(t)$ satisfies: $y_1(t) = 3x_1(t^3)$

Response $y_2(t)$ to $x_2(t)$ satisfies: $y_2(t) = 3x_2(t^3)$

Multiplying by arbitrary constants α_1 and $\alpha_2 \in \mathbb{R}$ and then adding them together yields:

$$\alpha_1 y_1(t) + \alpha_2 y_2(t) = \alpha_1 3x_1(t^3) + \alpha_2 3x_2(t^3)$$
$$\Rightarrow \quad \big(\alpha_1 y_1(t) + \alpha_2 y_2(t)\big) = 3\big(\alpha_1 x_1(t^3) + \alpha_2 x_2(t^3)\big)$$

Hence, the response to the input $\alpha_1 x_1(t) + \alpha_2 x_2(t)$ equals $\alpha_1 y_1(t) + \alpha_2 y_2(t)$. The superposition principle, (16.5), holds, the system is linear.

(c) This system is non-causal:
Take for instance $t = 2$ s: $y(2) = 3x(2^3) = 3x(8)$. We see that the system output at $t = 2$ s depends on the system input at $t = 8$ s. In other words, for $t > 1$ s, the system output depends on *future* values of the input.

$\mathcal{S}$.16-5

(a) This system is dynamic, time-invariant, linear and causal. Its order is 3 (take derivative, because of integral on output).

(b) This system is dynamic, time-invariant, nonlinear, because of the $y^2(t)$-term, and causal. Its order is 3.

(c) This system is dynamic, time-varying, because of the second coefficient on the left-hand side, $2t$, linear and causal. Its order is 2.

(d) This system is dynamic, time-varying, because of the second coefficient on the left-hand side, $2t$, nonlinear, because of the $y(t)\frac{\mathrm{d}^2y(t)}{\mathrm{d}t^2}$-term (see also Solution $\mathcal{S}$.16-1(d)), and causal. Its order is 2.

(e) This system is dynamic, because of the term $x(t^2)$, time-varying (see also Solution $\mathcal{S}$.16-4(a)), linear (see also Solution $\mathcal{S}$.16-4(b)), and non-causal, because of the term $x(t^2)$ (see also Solution $\mathcal{S}$.16-4(c)). Its order is zero (no integrals, no differential terms).

(f) This system is dynamic, time-varying, because of the second coefficient on the left-hand side, $2t$, nonlinear, because of the $y(t)\frac{\mathrm{d}^2y(t)}{\mathrm{d}t^2}$-term (see also Solution $\mathcal{S}$.16-1(d)), and non-causal, because of the $x(2t)$-term (see also Solution $\mathcal{S}$.16-4(c)). Its order is 2.

	System					
Property	(a)	(b)	(c)	(d)	(e)	(f)
(1) Dynamic	X	X	X	X	X	X
(2) Time-invariant	X	X				
(3) Linear	X		X		X	
(4) Causal	X	X	X	X		
(5) Order	3	3	2	2	0	2

$\mathcal{S}$.16-6

(a) This system is dynamic (output depends on past value of the input), time-invariant, linear and causal. Its order is zero (no integrals, no differential terms).

(b) This system is instantaneous (output depends on input at the present time t only), time-invariant, nonlinear, because of both the $x^{\frac{1}{2}}(t)$-term ($\sqrt{x(t)}$) and the term $+\frac{1}{2}$ on the right-hand side, and causal. Its order is zero (no integrals, no differential terms).

(c) This system is dynamic, time-varying, because of the second coefficient on the left-hand side, $\sqrt{t}$, linear and causal. Its order is 1.

(d) This system is dynamic, time-varying, because of the second coefficient on the left-hand side, $\frac{1}{t}$, linear and non-causal, because of the term $x(t+1)$. Its order is 2.

(e) This system is dynamic (output depends on past values of the input), time-invariant, linear and causal. Its order is 1 (take derivative, because of integral on input).

(f) This system is dynamic, time-invariant, nonlinear, because of the term $y(t)\frac{\mathrm{d}y(t)}{\mathrm{d}t}$, and causal. Its order is 4 (take derivative, because of integral on output).

	System					
Property	(a)	(b)	(c)	(d)	(e)	(f)
(1) Dynamic	X		X	X	X	X
(2) Time-invariant	X	X			X	X
(3) Linear	X		X	X	X	
(4) Causal	X	X	X		X	X
(5) Order	0	0	1	2	1	4

S.16-7

(a) This system is *not* dynamic, but instantaneous: the system output at a particular time t is a function of the system input at only that same particular time t, and does not depend on past or future values of the input signal. Hence, although the system is time-varying (see (b)), it is still instantaneous.

(b) This system is *not* time-invariant. Suppose input $x(t)$ equals 1 at all times, then $y(t) = 2\cos(25\pi t)$, which depends on time. Hence, this system is time-varying.
Better: the response to input signal $x_1(t)$ equals $y_1(t) = 2x_1(t)\cos(25\pi t)$. The response to a delayed input signal $x_2(t) = x_1(t-\tau)$ equals $y_2(t) = 2x_2(t)\cos(25\pi t) = 2x_1(t-\tau)\cos(25\pi t)$. Delaying the output $y_1(t)$ with τ yields: $y_1(t-\tau) = 2x_1(t-\tau)\cos\big(25\pi(t-\tau)\big)$. Clearly, $y_1(t-\tau) \neq y_2(t)$, they are not the same: the system is therefore time-varying.

(c) This system is linear:
Response $y_1(t)$ to $x_1(t)$ satisfies: $y_1(t) = 2x_1(t)\cos(25\pi t)$
Response $y_2(t)$ to $x_2(t)$ satisfies: $y_2(t) = 2x_2(t)\cos(25\pi t)$
Multiplying by arbitrary constants α_1 and $\alpha_2 \in \mathbb{R}$ and then adding them together yields:
$$\alpha_1 y_1(t) + \alpha_2 y_2(t) = \alpha_1 2x_1(t)\cos(25\pi t) + \alpha_2 2x_2(t)\cos(25\pi t)$$
$$\Rightarrow \quad (\alpha_1 y_1(t) + \alpha_2 y_2(t)) = (\alpha_1 x_1(t) + \alpha_2 x_2(t))\, 2\cos(25\pi t)$$
Hence, the response to the input $\alpha_1 x_1(t) + \alpha_2 x_2(t)$ equals $\alpha_1 y_1(t) + \alpha_2 y_2(t)$. The superposition principle, (16.5), holds, the system is linear.

(d) This system is causal, as the output does not depend on future values of the input.

S | 16.2 System response

S.16-8

The input-output relation of the linear time-invariant system is represented by $y(t) = \mathcal{H}\{x(t)\}$, (16.1). Substituting the expression for $x(t)$ given in the hint yields:
$$y(t) = \mathcal{H}\Big\{\int_{-\infty}^{\infty} x(\lambda)\,\delta(t-\lambda)\,\mathrm{d}\lambda\Big\}, \quad \text{which is} \quad y(t) = \int_{-\infty}^{\infty} x(\lambda)\,\mathcal{H}\{\delta(t-\lambda)\}\,\mathrm{d}\lambda$$
as a result of linearity (and the Dirac function being the only element dependent on time t). With time invariance of the system we have $h(t) = \mathcal{H}\{\delta(t)\}$ and $h(t-\lambda) = \mathcal{H}\{\delta(t-\lambda)\}$, hence, the above output becomes:
$$y(t) = \int_{-\infty}^{\infty} x(\lambda)h(t-\lambda)\,\mathrm{d}\lambda, \qquad \text{which is} \qquad y(t) = x(t) * h(t)$$
qed

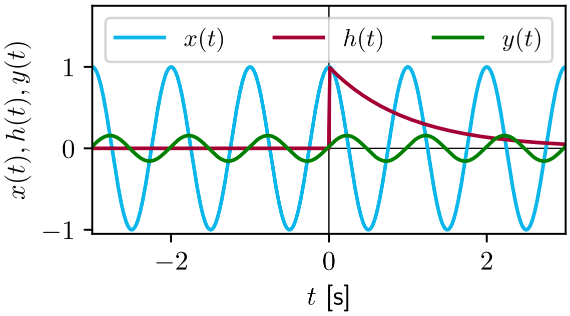
S | 16.c Computer exercises

S.16-9

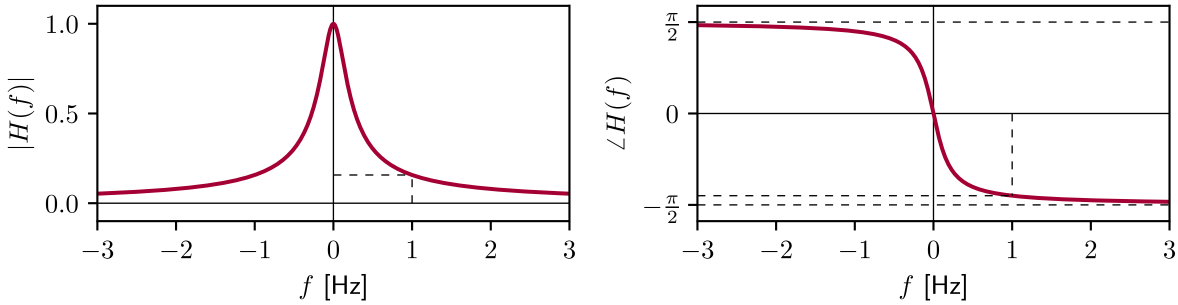
The output $y(t)$ of an LTI system (in general) to a cosine input $x(t)$, as in (16.12), with $A = 1$ and $\varphi = 0$, through convolution $y(t) = x(t) * h(t)$, is given by (16.14), and in this case becomes:
$$y(t) = |H(f_0)|\cos\big(2\pi f_0 t + \theta(f_0)\big) \qquad \text{with} \qquad H(f) = \frac{\alpha}{\alpha + j2\pi f} \qquad (16.10)$$
With $\alpha = 1$ and $f_0 = 1$ Hz, we have:
$$|H(f_0)| = \frac{|\alpha|}{\sqrt{\alpha^2 + (2\pi f_0)^2}} = 0.16$$
$\theta(f_0) = \angle H(f_0) = \arctan\Big(-\frac{2\pi f_0}{\alpha}\Big) = -1.41$ rad, which comes close to $-\frac{\pi}{2}$ rad.

This means that the amplitude of the output is 0.16, compared to 1 on the input, hence, the cosine is seriously attenuated. The phase change is almost $-\frac{\pi}{2}$, meaning that the cosine gets time delayed by almost a quarter of a period, in this case 0.25 s.

Input signal $x(t)$, impulse response function $h(t)$ and output $y(t)$ are shown on the right. Output $y(t)$ in green is the steady-state response of the LTI system to the given cosine on the input in blue. The frequency of the cosine on the output and input are the same!



The figure below shows the characteristics of the frequency response, i.e., $|H(f)|$ at left, and $\angle H(f)$ at right, with $f_0 = 1$ Hz indicated.



Basic Python code:

```
import numpy as np
from matplotlib import pyplot as plt

burgundy = [165/255,0/255,52/255]   # burgundy color

# demonstrate a cosine (harmonic) going through an LTI system
# (specifically with h(t) = a exp(-at) u(t))

# continuous time approximated by Delta_t = 0.01 s

# (original) 'infinite' time window length
f0  = 1                              # cosine frequency [Hz]
a   = 1                              # time constant of 1st-order ODE system
t   = np.linspace(-10, 10, 2001)     # time steps 0.01 s from -10 s to 10 s
xt  = np.cos(2*np.pi*f0*t)       # input signal (cosine, zero initial phase)
ht  = a * np.exp(-a*t) * np.heaviside(t, 0)# impulse response function h(t)

# output through numerical convolution (as a check)
# discretize convolution integral: multiply by 0.01(correct for step size)
yt_conv = np.convolve(xt, ht, 'same') * 0.01
# yt_conv differs from yt below by 0.025 in the beginning, falls off, and
# later the difference oscillates up to 0.005

# output through equation in companion book, Theory
Hf0 = a / (a + 2j*np.pi*f0)     # frequency response function H(f) at f = f0
yt_eq = np.abs(Hf0) * np.cos(2*np.pi*f0*t + np.angle(Hf0))

# plot of x(t), h(t), y(t)
plt.figure()
```

```
plt.plot(t, xt, color='b', label='$x(t)$')
plt.plot(t, ht, color=burgundy, label='$h(t)$')
plt.plot(t, yt_eq, color='g', label='$y(t)$')

# frequency response
f = np.linspace(-3.0, 3.0, 601)  # frequency steps 0.01 Hz from -3 to 3 Hz
Hf = a / (a + 2j*np.pi*f)        # frequency response function H(f)

# plot of frequency response
plt.figure()
plt.subplot(3,2,1)
plt.plot(f, np.abs(Hf), color=burgundy)
plt.xlabel('$f$ [Hz]'); plt.ylabel('$|H(f)|$')
plt.subplot(3,2,2)
plt.plot(f, np.angle(Hf), color=burgundy)
plt.xlabel('$f$ [Hz]'); plt.ylabel('$\\angle H(f)$')
```

S.16-10 The output $y(t)$ of an LTI system (in general) to a cosine input $x(t)$, as in (16.12), is given by (16.14). In this exercise, we use *linearity* (superposition): the (steady-state) output of an LTI system to a sum of cosines on the input simply is the sum of the outputs to the individual cosines:

$$y(t) = |H(f_1)| \cos\big(2\pi f_1 t + \theta(f_1)\big) + |H(f_2)| \cos\big(2\pi f_2 t + \theta(f_2)\big)$$

$$\text{with} \qquad H(f) = \frac{\alpha}{\alpha + j2\pi f} \qquad (16.10)$$

With $\alpha = 1$ and $f_1 = 1$ Hz we have: $|H(f_1)| = 0.16$ and $\theta(f_1) = \angle H(f_1) = -1.41$

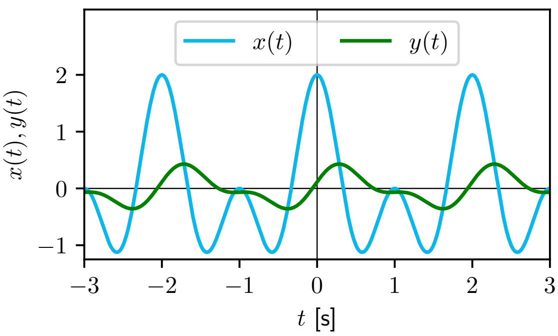
With $\alpha = 1$ and $f_2 = \frac{1}{2}$ Hz we have: $|H(f_2)| = 0.30$ and $\theta(f_2) = \angle H(f_2) = -1.26$

$|H(f)|$ and $\theta(f) = \angle H(f)$ are shown in the figure at the bottom of Solution S.16-9.

Key to note here is that harmonics at *different frequencies* experience a different attenuation, and a different phase shift (and hence time delay) in the LTI system. The cosine with the smaller frequency f_2 gets less attenuated and a smaller phase shift than the cosine with frequency f_1.

The consequence is that the signal gets *distorted*. The different components of the signal experience different behavior, causing deformation of the signal.

As can be seen in the figure on the right, the output signal $y(t)$ in green no longer simply is an attenuated and delayed copy of the input signal $x(t)$ in blue. Instead, the LTI system has 'modified' the signal.



Basic Python code:

```
import numpy as np
from matplotlib import pyplot as plt

# demonstrate two cosines (harmonics) going through an LTI system
# (specifically with h(t) = a exp(-at) u(t))
# and due to different attenuation and phase change (delay), at different
# frequencies, the output signal is distorted compared to the input
```

```
# continuous time approximated by Delta_t = 0.01 s

# (original) 'infinite' time window length
f1 = 1                              # cosine carrier frequency 1 [Hz]
f2 = 0.5                            # cosine carrier frequency 2 [Hz]
a  = 1                              # time constant of 1st-order ODE system
t  = np.linspace(-10, 10, 2001)     # time steps 0.01 s from -10 s to 10 s
x1t = np.cos(2*np.pi*f1*t)    # input signal 1 (cosine, zero initial phase)
x2t = np.cos(2*np.pi*f2*t)    # input signal 2 (cosine, zero initial phase)
ht = a * np.exp(-a*t) * np.heaviside(t, 0)# impulse response function h(t)

# output through equation in companion book, Theory, and superposition
Hf1 = a / (a + 2j*np.pi*f1)         # frequency response function H(f)
Hf2 = a / (a + 2j*np.pi*f2)
y1t = np.abs(Hf1) * np.cos(2*np.pi*f1*t + np.angle(Hf1))        # output 1
y2t = np.abs(Hf2) * np.cos(2*np.pi*f2*t + np.angle(Hf2))        # output 2

# plot of x(t) and y(t)
plt.figure()
plt.plot(t, (x1t+x2t), color='b', label='$x(t)$')
plt.plot(t, (y1t+y2t), color='g', label='$y(t)$')
plt.xlabel('$t$ [s]'); plt.ylabel('$x(t), y(t)$')
```

17

Measuring signals

Exercises

e | 17.1 Sensor

In Section 17.1 the (classic) accelerometer was introduced, Figure 17.1. The accelerometer was put on a horizontal surface, aligned with its input-output axis, sensing (input) acceleration $\ddot{x}(t)$ of the housing (in [m/s^2]) through measuring the *displacement* $y(t)$ of the proof mass which can move only along the input-output axis. e.17-1

In this exercise, we assume that the accelerometer is perfectly linear, has no bias, has a sensitivity of 1 and no measurement noise exists. In addition, in all questions, we assume the accelerometer to be at rest at $t = 0$ s.

(a) While keeping the accelerometer perfectly horizontal, at $t = 0$ s it is accelerated with 10 m/s^2 to the right, $\ddot{x}(t) = 10u(t)$, with $u(t)$ the unit step function, (B.5). When at steady state, what would be the position of the proof mass within the housing relative to its neutral position? What would be the output of the accelerometer?

(b) The accelerometer is turned 90 degrees in counterclockwise direction (i.e., its input-output axis is now perfectly vertical, with the positive x axis upwards) and put on a horizontal table, standing perfectly still, $\ddot{x}(t) = 0$. What would be the position of the proof mass within the housing relative to its neutral position? What would be the output of the accelerometer?

(c) The accelerometer, with its input-output axis remaining perfectly vertical, as in (b), is brought into a vacuum drop tower. At $t = 0$ s, the accelerometer is dropped, it falls down, $\ddot{x}(t) = -9.81u(t)$. When at steady state, what would be the position of the proof mass within the housing relative to its neutral position? What would be the output of the accelerometer?

(d) When operating the accelerometer on Earth (or another planet), and also considering orientations of the accelerometer other than perfectly horizontal (as in (a)) or vertical (as in (b)), what *is* the input to this sensor?

(e) When operating the accelerometer in interstellar space (assume zero gravity), what is the input to this sensor?

(f) From your answers to the questions above, decide on whether 'accelerometer' is a proper name for this device.

e| 17.c Computer exercises

e.17-2

A geophone, a primary sensor in seismic surveys (to measure ground movement), is a mass-spring-damper system. A magnet is fixed to the sensor housing, and a copper wire coil is suspended by springs. The wire coil acts as a proof mass. When the sensor is exposed to motion, the wire coil will try to maintain its position in inertial space and move with respect to the sensor housing (and the magnet). Thereby it will experience a change in magnetic flux; consequently, an electrical voltage is induced. The resulting signal is proportional to the time rate of change of the magnetic flux; hence, the output voltage is proportional to the *velocity* of the coil with respect to the magnet.

(a) Analytically derive frequency response function $H(\omega)$ for a geophone, starting from the second-order differential equation (17.1):

$$\ddot{y}(t) + 2\zeta\omega_0\,\dot{y}(t) + \omega_0^2\,y(t) = -\ddot{x}(t)$$

with ζ the (dimensionless) damping ratio and ω_0 the undamped natural angular frequency in [rad/s] (and a sensor layout similar to Figure 17.1 at left).
We aim to solve the differential equation in terms of input velocity $\dot{x}(t)$ and output velocity $\dot{y}(t)$.
Hint: use the differentiation Fourier transform theorem (6.10) and integration theorem (6.11): $x(t) \overset{\mathcal{F}}{\leftrightarrow} X(f)$, then $\frac{\mathrm{d}x(t)}{\mathrm{d}t} \overset{\mathcal{F}}{\leftrightarrow} (j2\pi f)X(f)$, with $\omega = 2\pi f$, and $\int_{-\infty}^{t} x(\tau)\,\mathrm{d}\tau \overset{\mathcal{F}}{\leftrightarrow} \frac{1}{j2\pi f}X(f)$, which holds under the condition that $X(0) = 0$, cf. (5.8).

(b) Create the Bode plot of frequency response function $H(\omega)$ for a geophone for $\zeta = 0.2$ and $\omega_0 = 1$ rad/s.

Solutions

s|17.1 Sensor

s.17-1

(a) With this input and horizontal orientation of the accelerometer, the proof mass would be positioned to the left of its neutral position in its housing, $y(t) < 0$, at a fixed position. The output of the accelerometer would be $+10$ V, indicating a 10 m/s^2 acceleration to the right.

(b) With this input and vertical orientation of the accelerometer, the proof mass would be positioned to the left (=down) of its neutral position in its housing, $y(t) < 0$, at a fixed position. The output of the accelerometer would be $+9.81$ V, indicating a 9.81 m/s^2 acceleration (also referred to as 1 'g') upwards.

(c) With this input and vertical orientation of the accelerometer, the proof mass would be positioned at its neutral position in its housing, $y(t) = 0$. The output of the accelerometer would be 0 V, indicating a 0 m/s^2 acceleration. Both the housing and the proof mass experience the same acceleration.

(d) On Earth, with a vertical orientation of the accelerometer, the accelerometer senses the inertial acceleration $\ddot{x}(t)$ of the accelerometer housing minus gravity $g \approx 9.81$ m/s^2, that is, its output is proportional to $\ddot{x}(t) - (-g) = \ddot{x}(t) + g$ (recall that positive accelerations are defined to the right (=up) along the accelerometer's positive x (=input-output) axis, see Figure 17.1). Only when lying perfectly horizontal, the accelerometer only senses the inertial acceleration. In the general case, defining the orientation of the accelerometer's positive x axis relative to the local horizontal using angle φ, the accelerometer output is proportional to $\ddot{x}(t) + g \sin\varphi$.

(e) In interstellar space, assuming zero gravity, the accelerometer senses the inertial acceleration $\ddot{x}(t)$, no matter what orientation it has.

(f) The name is dubious. After all, in (b), the accelerometer was standing perfectly still on a surface, while its output was 9.81 V. And in (c), the accelerometer was in free fall, accelerating with $\ddot{x}(t) = -9.81$ m/s^2 $= -g$ towards the Earth surface, while its output was 0 V.

s|17.c Computer exercises

s.17-2

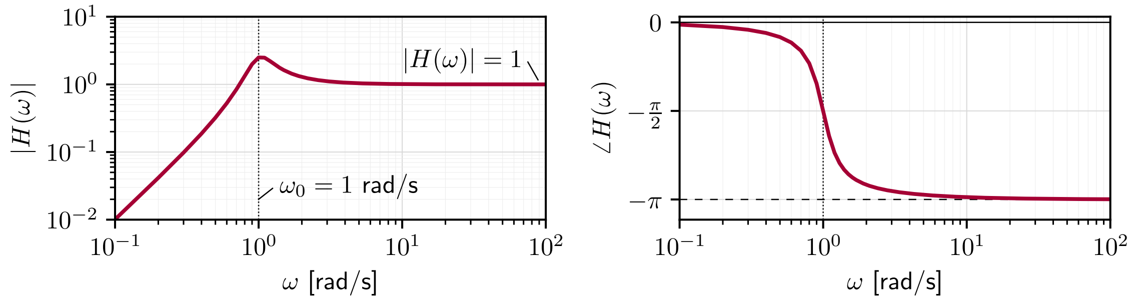
(a) The Fourier transforms of the input and output velocity, $\dot{x}(t)$ and $\dot{y}(t)$, respectively, are denoted by $\dot{X}(\omega)$ and $\dot{Y}(\omega)$. Transforming the given second-order differential equation to the frequency domain yields:

$$j\omega\dot{Y}(\omega) + 2\zeta\omega_0\dot{Y}(\omega) + \omega_0^2\frac{1}{j\omega}\dot{Y}(\omega) = -j\omega\dot{X}(\omega)$$

The frequency response function $H(\omega)$ is then found as:

$$H(\omega) = \frac{\dot{Y}(\omega)}{\dot{X}(\omega)} = \frac{\omega^2}{\omega_0^2 + 2j\zeta\omega_0\omega - \omega^2} \qquad \text{relating output velocity to input velocity.}$$

(b) The figure on the next page shows the Bode plot of frequency response function $H(\omega)$ for a geophone, with magnitude response $|H(\omega)|$ at left and phase response $\angle H(\omega)$ at right for damping ratio $\zeta = 0.2$ and undamped natural frequency $\omega_0 = 10^0 = 1$ rad/s.



Without damping the geophone would go in resonance at the undamped natural frequency $\omega = \omega_0$, indicated by the black dashed vertical line. The application range of a geophone, in terms of frequency, lies beyond the undamped natural frequency ω_0, that is $\omega > \omega_0$, then $H(\omega) \uparrow -1$ (negative, as for an input velocity to the right, the proof mass will respond to the left) and $|H(\omega)| \downarrow 1$. In this regime the wire coil will respond one-to-one to input velocity. Typical values in practice are $f_0 = 10$ Hz and $\zeta = 0.707$.

Basic Python code:

```
import numpy as np
from matplotlib import pyplot as plt

burgundy = [165/255,0/255,52/255]      # burgundy color

# plot frequency response function (FRF) of a geophone:
# amplitude |H(w)| and phase arg(H(w)), as function of angular frequency w

# input ground velocity (positive direction up)
# output proof mass velocity (positive direction up)

N    = 1000
w    = np.linspace(0.1, 100.0, N)      # steps in angular frequency [rad/s]
w0   = 1.0                  # (undamped) natural frequency omega0 = 1 rad/s
zeta = 0.2                             # damping ratio [-]

x1  = np.power(w,2)                    # numerator
x2  = np.power(w0,2)*np.ones(N) + 2j*zeta*w0*w - np.power(w,2)# denominator
Hw  = np.divide(x1,x2)                 # frequency response function

# plot frequency response
plt.figure()
plt.subplot(3,2,1)                     # magnitude response
plt.loglog(w, np.abs(Hw), color=burgundy, linewidth=1.5)
plt.xlabel('$\omega$ [rad/s]'); plt.ylabel('$|H(\omega)|$')

plt.subplot(3,2,2)                     # phase response
plt.semilogx(w, np.angle(Hw), color=burgundy, linewidth=1.5)
plt.xlabel('$\omega$ [rad/s]'); plt.ylabel('$\\angle H(\omega)$')
```

18

Filters

Exercises

e| 18.1 Ideal filters

e.18-1

Consider input voltage signal

$$x(t) = 12\,\mathrm{sinc}(3t)\cos^2(12\pi t)$$

which is fed through an ideal filter, resulting in output signal

$$y(t) = 24\,\mathrm{sinc}(3t)\cos(24\pi t)$$

(a) Determine and sketch frequency response function $H(f)$ of the filter.

(b) Compute the energy of output signal $y(t)$.

(c) Are there any alternative frequency response functions $H(f)$ that you may choose to arrive at $y(t)$ from $x(t)$? Explain your answer.

e.18-2

A linear time-invariant system turns input signal

$$x(t) = 6\,\mathrm{sinc}(4t)\cos^2(7\pi t)$$

into output signal $y(t)$.

Determine and sketch frequency response function $H(f)$ of the system, such that the output signal will be:

(a) $y(t) = 6\,\mathrm{sinc}(4t)$

(b) $y(t) = 4\,\mathrm{sinc}(4t)\cos(14\pi t)$

e.18-3

A signal $x(t)$ is fed through an ideal bandpass filter with bandwidth $f_2 - f_1$, centered at frequency f_{center}, and in the pass band a unit gain, i.e., $|H(f)| = 1$, resulting in output signal $y(t)$. Both $x(t)$ and $y(t)$ are voltage signals.

For the two separate input signals $x(t)$ defined below, what is the power of output signal $y(t)$ for the specified filter characteristics?

(a)	$x(t) = 2 + 10\sin^2(4\pi t) + 8\cos\left(20\pi t + \frac{\pi}{12}\right)$	$f_2 - f_1 = 4$ Hz	$f_{\text{center}} = 5$ Hz
(b)	$x(t) = 10\cos^2\left(20\pi t + \frac{\pi}{3}\right) - 5\sin\left(100\pi t + \frac{\pi}{2}\right)$	$f_2 - f_1 = 10$ Hz	$f_{\text{center}} = 35$ Hz

e.18-4 Signal

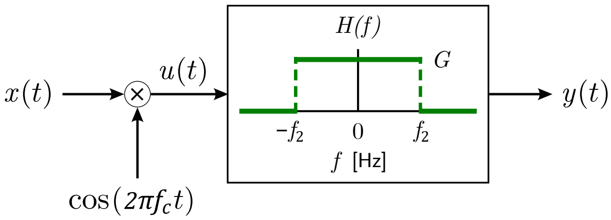
$$u(t) = x(t)\cos(2\pi f_c t)$$

is fed through an ideal low-pass filter with frequency response function $H(f)$, resulting in output signal $y(t)$, as illustrated below on the right.
The Fourier transforms of signals $x(t)$ and $y(t)$ are defined as

$$X(f) = \cos(\pi f)\,\Pi(f)$$

and

$$Y(f) = 2X(f+f_0) + 2X(f-f_0)$$



(a) Obtain time-domain signal $x(t)$.

(b) Obtain time-domain signal $y(t)$.

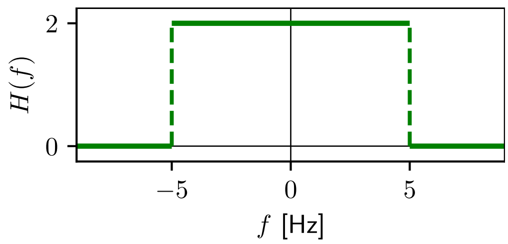
(c) Determine frequency f_c of signal $u(t) = x(t)\cos(2\pi f_c t)$ as well as gain G and bandwidth frequency f_2 of frequency response function $H(f)$ of the low-pass filter.

e.18-5 Obtain impulse response function $h_{HP}(t)$ of the ideal high-pass filter with frequency response function

$$H_{HP}(f) = G\left(1 - \Pi\left(\frac{f}{2B}\right)\right)e^{-j2\pi f t_0}$$

e.18-6 An unknown signal $x(t)$ is fed through an ideal low-pass filter, resulting in output signal $y(t)$. Frequency response function $H(f)$ of the filter is illustrated below at right. Both $x(t)$ and $y(t)$ are voltage signals. The output signal is defined by:

$$y(t) = 2 + 10\,\mathrm{sinc}(10t)$$



Which of the following functions describes input signal $x(t)$ correctly?

A. $x(t) = 1 + 4\,\mathrm{sinc}(4t)\cos(10\pi t)$

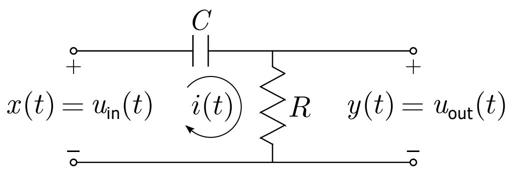
B. $x(t) = 1 + 10\,\mathrm{sinc}(10t)\cos(10\pi t)$

C. $x(t) = 2 + 20\,\mathrm{sinc}(20t)$

D. None of the above functions is correct.

e| 18.2 Practical filters

e.18-7 A high-pass filter constructed from a resistor-capacitor (RC) network is shown on the right. According to Ohm's law, output voltage $u_{\text{out}}(t)$, current $i(t)$ and resistance R are related through:



$$u_{\text{out}}(t) = R\,i(t)$$

where current $i(t)$ is proportional to the time rate of change of the voltage difference across the capacitor with capacitance C:

$$i(t) = C\frac{\mathrm{d}}{\mathrm{d}t}\big(u_{\text{in}}(t) - u_{\text{out}}(t)\big)$$

(a) Derive the ordinary differential equation for this system from Ohm's law.
(b) Obtain frequency response function $H(f)$ of the RC high-pass filter.
(c) Under what conditions will the filter perform as an ideal differentiator defined by the input-output relation

$$y(t) = A\frac{\mathrm{d}}{\mathrm{d}t}x(t)$$

Hint 1: a differentiator has frequency response function $H(f) = j2\pi f$.
Hint 2: in these types of questions, it helps to create a Bode plot of $H(f)$ for different values of cut-off frequency $f_3 = \frac{1}{2\pi RC}$, e.g., 0.08, 0.16, 0.32 and 1.59 Hz. Exercise *e*.18-8 shows how to do this.

e| 18.c Computer exercises

e.18-8

We return to Exercise *e*.18-7, where it was suggested to create a Bode plot of $H(f)$ for different values of the cut-off frequency f_3.

(a) For $f_3 = 1.59$ Hz, create a Bode plot which shows the amplitude response $|H(f)|$ and phase response $\angle H(f)$ for the frequency range from 0.001 Hz to 50 Hz.
(b) Augment your script such that it can show the Bode plot of $H(f)$ for four values of f_3 (0.08 Hz, 0.16 Hz, 0.32 Hz and 1.59 Hz) in one plot.

e.18-9

In this exercise, we demonstrate the effect that filtering has on a signal which consists of three cosines at different frequencies. This analysis is done in continuous time (infinite length), according to the exposition in Section 16.4: Frequency response function.

Signal $x(t)$ consists of three cosines, all with unit amplitude $A = 1$ and zero phase $\varphi = 0$, with frequencies $f_1 = 0.1$ Hz, $f_2 = 0.2$ Hz and $f_3 = 0.3$ Hz.
The signal is fed through a first-order Butterworth low-pass filter, with frequency response function

$$H(f) = \frac{1}{1 + j2\pi f} \tag{18.1}$$

resulting in output signal $y(t)$.

(a) Create and plot input signal $x(t)$ for $t \in [-5, 5]$ s.
(b) Construct and plot (steady-state) output signal $y(t)$.
(c) Compare output signal $y(t)$ with input signal $x(t)$.

e.18-10

In Exercise *e*.18-9, we explored the effects of a first-order Butterworth low-pass filter. In this exercise, we examine the frequency response of a second-order Butterworth low-pass filter:

$$H(f) = \frac{1}{1 + j\sqrt{2}\frac{f}{f_3} - \left(\frac{f}{f_3}\right)^2}$$

where f_3 is the cut-off frequency of the filter in [Hz].

(a) Plot amplitude response $|H(f)|$ and phase response $\angle H(f)$ of the filter for the frequency range from -1.5 Hz to $+1.5$ Hz. Set cut-off frequency f_3 to $\frac{1}{2\pi}$ Hz.
(b) Compare the filter operation of a second-order Butterworth low-pass filter with that of a first-order one using Figure 18.8.

e.18-11 In this exercise, we apply a first-order Butterworth low-pass filter in discrete time (and on a finite-length signal). We consider a $T = 10$ s data record of the signal with the three cosines as in Exercise e.18-9 (though time-shifted here by 5 s):

$$x(t) = \sum_{i=1}^{3} \cos(2\pi f_i(t-5)) \qquad \text{for} \quad t \in [0, T)$$

with a sampling interval of $\Delta t = 0.01$ s ($f_s = 100$ Hz).

As outlined in Section 18.3, the discrete-time filter impulse response h_n can be obtained by sampling the continuous-time impulse response $h(t)$, in this case given by (18.7), as shown in (18.9). The resulting discrete-time filter impulse response is shown in Figure 18.12 at left. The output sequence can be computed through discrete convolution: $y_n = x_n * h_n$, cf. Appendix F. In practice, a filter operation is commonly implemented based on a difference equation.

(a) Create and plot input signal $x(t)$ in discrete time, i.e., sequence x_n. Show the samples as a function of the 'restored' continuous time t in [s] on the horizontal axis (hereafter referred to as 'time t').

(b) Apply a digital first-order Butterworth low-pass filter to (input) sequence x_n, and plot output sequence y_n as a function of time t. The cut-off frequency of the filter is taken as $f_3 = \frac{1}{2\pi}$ Hz (see Section 18.2.1).

Hint 1: in Python, one can design a Butterworth filter using the function `scipy.signal.butter`, which sets and returns the filter coefficients. Do not forget to include the sampling frequency as an argument in the function.

Hint 2: in Python, one can apply a digital filter to a data sequence using the function `scipy.signal.lfilter`, which accepts the filter coefficients returned by `scipy.signal.butter` as part of its inputs.

(c) Apply the same first-order Butterworth low-pass filter as in (b) to input sequence x_n, but this time use the Python function `scipy.signal.filtfilt` (with default options) instead of `scipy.signal.lfilter`, and again plot output sequence y_n as a function of time t.

(d) Discuss how the two output sequences y_n obtained using `scipy.signal.lfilter` and `scipy.signal.filtfilt`, respectively, differ from input sequence x_n and from each other.

e.18-12 Note that this is an analytical, by-hand exercise that examines the underlying algorithm of a Python function, rather than a computer-based exercise.

In Exercise e.18-11, the Python function `scipy.signal.filtfilt` was introduced. This function performs a filter operation on an input signal, by applying the (same) filter *twice*, once forward and once backward. The particular benefit of this approach is that the entire operation is zero phase. In this way, the filter does not cause any phase shift to the signal, and thus also no time delay at all.

In the time domain, the procedure can be cast as follows (for convenience formulated in continuous time):

1. Input signal $x(t)$ is convolved with the impulse response of the filter $h(t)$.
2. Output signal $y(t)$ is then time-reversed.
3. The time-reversed output is convolved with $h(t)$ another time.
4. The result is again time-reversed to arrive at the double-filtered signal, as a function of (forward) time.

Starting from Fourier transform $X(f)$ of input signal $x(t)$ and Fourier transform $H(f)$ of filter impulse response $h(t)$, with both signal $x(t)$ and impulse response $h(t)$ being real functions:

(a) Derive the Fourier transform of the output signal of the above double-filtering procedure.

(b) Show that indeed this operation is zero phase.

e.18-13

In Section 5.8, we have learned that the extremely 'narrow' Dirac delta function in time, $x(t) = \delta(t)$, when Fourier-transformed, becomes extremely 'broad' in frequency, where $X(f) = 1 \; \forall \; f$. In this exercise, we explore what happens to the Dirac delta function in the time domain when we apply a filter to it, i.e., when we remove part of the signal's spectrum in the frequency domain.

Consider a signal $x(t)$, which is sampled with sampling interval $\Delta t = 0.001$ s, starting at $t = 0.000$ s, for time duration T, such that a total of $N = 256$ samples are obtained. All samples have a value of 0, except for sample number 128, which has a value equal to the sampling frequency f_s. Hence, at this sample number a Dirac pulse occurs. Remember from Exercises *e*.12-5 and *e*.12-6 that a discrete sample represents a time interval of $\Delta t = \frac{1}{f_s}$, and by having the amplitude equal to f_s, we ensure that the area of the Dirac pulse equals 1.

(a) Create and plot discrete-time sequence x_n. Show the samples as a function of time t.
Hint: in Python, sample number 128 has index 127.

(b) Compute the DFT of x_n and plot magnitude spectrum $|X_k|$ as a function of the 'restored' continuous frequency f in [Hz] on the horizontal axis (hereafter referred to as 'frequency f'), symmetric about $f = 0$.

(c) Apply a (digital) second-order Butterworth low-pass filter to sequence x_n such that frequencies above 100 Hz are filtered out.
Hint: in Python, use the functions `scipy.signal.butter` and `scipy.signal.filtfilt`, which you also used in Exercise *e*.18-11. Do not forget to include the sampling frequency as an argument in the function `scipy.signal.butter`.

(d) Plot output sequence y_n in the same graph as input sequence x_n.

(e) Compute the DFT of y_n and plot magnitude spectrum $|Y_k|$ in the same graph as magnitude spectrum $|X_k|$.

(f) Again apply a second-order Butterworth low-pass filter to the (original) sequence x_n, but this time with a cut-off frequency of 200 Hz.

(g) To demonstrate how filtering with different cut-off frequencies affects the width of the band-limited unit impulse, plot the 'new' output sequence y_n as well as its magnitude spectrum $|Y_k|$ in the same graphs as the earlier plotted discrete-time sequences and magnitude spectra.

(h) Comment on the effects of the two filters, each with a different cut-off frequency, on the amplitude and shape of the unit impulse signal.

e.18-14

A main reason for filtering to work is that, in practice, many signals are overlapping in time (occurring at the same time), but are quite well separated in the frequency domain. In this exercise, we look at two overlapping sinusoids with different frequencies, trying to filter out the high-frequency one, while keeping the low-frequency one.

Consider the following signal consisting of two sines with different frequencies and amplitudes:

$$x(t) = \frac{1}{2}\sin(40\pi t) - \sin(2\pi t)$$

This signal is sampled with sampling interval $\Delta t = 0.001$ s, starting at $t = 0.000$ s, for time duration T, such that a total of $N = 2000$ samples are obtained.

(a) Create and plot discrete-time sequence x_n. Show the samples as a function of time t.

(b) Compute the DFT of x_n and plot magnitude spectrum $|X_k|$ as a function of frequency f, symmetric about $f = 0$.
Hint: multiply Python's FFT result by Δt and divide by T to maintain the analogy with the Fourier transform of continuous-time signal $x(t)$.

(c) Apply a (digital) second-order Butterworth low-pass filter to sequence x_n such that the *high-frequency* sine is filtered out. What value do you take for the cut-off frequency?

(d) Plot output sequence y_n in the same graph as input sequence x_n.

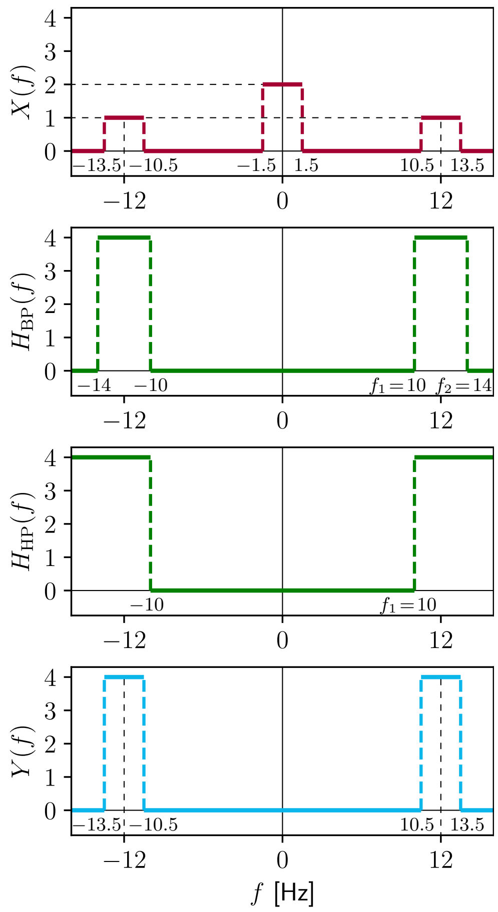
(e) Compute the DFT of y_n and plot magnitude spectrum $|Y_k|$ in the same graph as magnitude spectrum $|X_k|$.

(f) Test what happens to the filtered result when you choose a larger cut-off frequency for the filter, as well as when you choose a smaller one. Comment on your observations.

Solutions

s | **18.1** Ideal filters

s.18-1



(a) Using $\cos^2 u = \frac{1}{2}(1+\cos(2u))$, (A.4):

$$\begin{aligned} x(t) &= 12\,\mathrm{sinc}(3t)\cos^2(12\pi t) \\ &= 12\,\mathrm{sinc}(3t)\,\tfrac{1}{2}(1+\cos(24\pi t)) \\ &= 6\,\mathrm{sinc}(3t) + 6\,\mathrm{sinc}(3t)\cos(24\pi t) \end{aligned}$$

Using $\tau\,\mathrm{sinc}(\tau t) \overset{\mathcal{F}}{\leftrightarrow} \Pi\left(\frac{f}{\tau}\right)$, (6.13), in combination with the modulation theorem, (6.7), $x_1(t)\cos(2\pi f_0 t) \overset{\mathcal{F}}{\leftrightarrow} \frac{1}{2}X_1(f+f_0)+\frac{1}{2}X_1(f-f_0)$, we obtain for Fourier transform $X(f)$:

$$X(f) = 2\Pi\left(\frac{f}{3}\right) + \Pi\left(\frac{f+12}{3}\right) + \Pi\left(\frac{f-12}{3}\right)$$

$X(f)$ is shown on the right at the top.
When also using (6.7) and (6.13) to obtain $Y(f)$ for $y(t) = 24\,\mathrm{sinc}(3t)\cos(24\pi t)$, we get:

$$Y(f) = 4\Pi\left(\frac{f+12}{3}\right) + 4\Pi\left(\frac{f-12}{3}\right)$$

$Y(f)$ is shown on the right at the bottom.
Clearly, when looking at the illustrations of $X(f)$ and $Y(f)$, output $Y(f)$ can be obtained by filtering $X(f)$ with an ideal bandpass filter with frequency response function $H_{BP}(f)$ with an amplitude, or so-called gain, of 4, and which maintains the part of input signal $x(t)$ for absolute frequencies from $|f| = 10.5$ to $|f| = 13.5$ Hz, and filters out the part of the signal for absolute frequencies from $f = 0$ to $|f| = 1.5$ Hz. When setting bandwidth frequencies of $f_1 = 10$ Hz and $f_2 = 14$ Hz, we achieve this result. Hence, $H_{BP}(f) = 4\Pi\left(\frac{f+12}{4}\right)+4\Pi\left(\frac{f-12}{4}\right)$ as illustrated in the second row on the right, would work. Alternatively, an ideal high-pass filter with frequency response function $H_{HP}(f)$ with the same gain (of 4) and a bandwidth frequency $f_1 = 10$ Hz, would give the same output $Y(f)$ for the current input $X(f)$, as illustrated in the third row on the right.

(b) We compute the energy of output signal $y(t)$ in the frequency domain using Parseval's theorem, (5.19):

$$E_y = \int_{-\infty}^{\infty} |Y(f)|^2\,\mathrm{d}f = \int_{-13.5}^{-10.5} 4^2\,\mathrm{d}f + \int_{10.5}^{13.5} 4^2\,\mathrm{d}f = 96\text{ J}$$

Looking at the illustration of $Y(f)$ above, it can also almost directly be seen that:

$$E_y = 3\cdot 4^2 + 3\cdot 4^2 = 2\cdot 3\cdot 4^2 = 96\text{ J}$$

(c) This question was already partially answered in (a). Both an ideal bandpass and an ideal high-pass filter could work, with some flexibility in choosing the bandwidth frequency/frequencies of those filters. As long as (1) the filter gain is set to 4, (2) the filter lets input signal components pass for absolute frequencies from $f = 10.5$ to $f = 13.5$ Hz, and (3) the filter blocks input signal components for absolute frequencies from $f = 0$ to $f = 1.5$ Hz, output signal $y(t)$ will be as specified in the exercise, given input signal $x(t)$ as defined.

S.18-2

(a) Using $\cos^2 u = \frac{1}{2}(1+\cos(2u))$, (A.4):

$$\begin{aligned} x(t) &= 6\,\mathrm{sinc}(4t)\cos^2(7\pi t) \\ &= 6\,\mathrm{sinc}(4t)\,\tfrac{1}{2}\big(1+\cos(14\pi t)\big) \\ &= 3\,\mathrm{sinc}(4t) + 3\,\mathrm{sinc}(4t)\cos(14\pi t) \end{aligned}$$

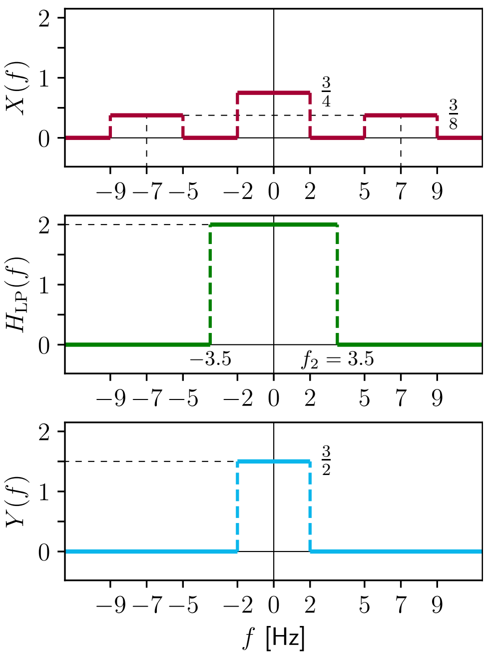
Using $\tau\,\mathrm{sinc}(\tau t) \overset{\mathcal{F}}{\leftrightarrow} \Pi\left(\frac{f}{\tau}\right)$, (6.13), in combination with the modulation theorem, (6.7), $x_1(t)\cos(2\pi f_0 t) \overset{\mathcal{F}}{\leftrightarrow} \frac{1}{2}X_1(f+f_0)+\frac{1}{2}X_1(f-f_0)$, we obtain for Fourier transform $X(f)$:

$$X(f) = \tfrac{3}{4}\Pi\left(\frac{f}{4}\right) + \tfrac{3}{8}\Pi\left(\frac{f+7}{4}\right) + \tfrac{3}{8}\Pi\left(\frac{f-7}{4}\right)$$

$X(f)$ is shown on the right at the top.
When also using (6.13) to obtain $Y(f)$ for $y(t) = 6\,\mathrm{sinc}(4t)$, we get:

$$Y(f) = \tfrac{3}{2}\Pi\left(\frac{f}{4}\right)$$

$Y(f)$ is shown on the right at the bottom.



Clearly, when looking at the illustrations of $X(f)$ and $Y(f)$, output $Y(f)$ can be obtained by filtering $X(f)$ with an ideal low-pass filter with frequency response function $H_{LP}(f)$ with an amplitude, or so-called gain, of 2, and that maintains the part of input signal $x(t)$ for absolute frequencies from $f = 0$ to $|f| = 2$ Hz, and that filters out the part of the signal for higher (absolute) frequencies. When setting a bandwidth frequency $f_2 = 3.5$ Hz, we achieve this result. Hence, $H_{LP}(f) = 2\Pi\left(\frac{f}{7}\right)$ as illustrated in the middle on the right, would work. However, note that setting bandwidth frequency f_2 to 3.5 Hz is rather arbitrary, as input $X(f)$ equals 0 between the absolute frequencies of 2 Hz and 5 Hz, so any bandwidth frequency f_2 within that range would lead to the requested output signal $y(t)$ when filtering the given input signal $x(t)$.

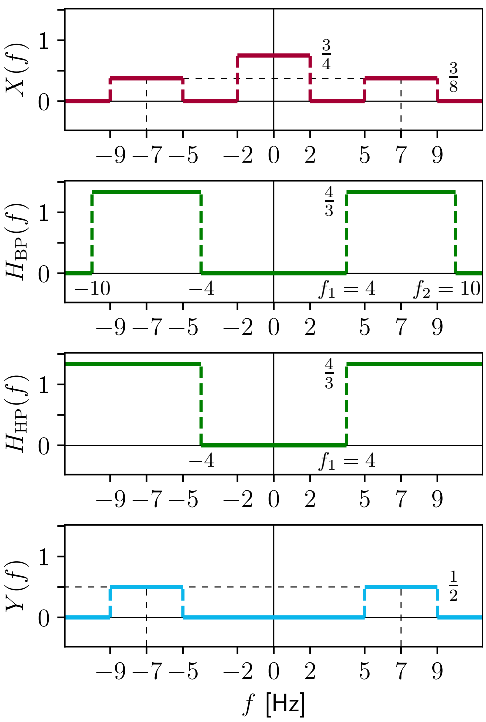
(b) Fourier transform $X(f)$ was already determined in (a):

$$X(f) = \tfrac{3}{4}\Pi\left(\frac{f}{4}\right) + \tfrac{3}{8}\Pi\left(\frac{f+7}{4}\right) + \tfrac{3}{8}\Pi\left(\frac{f-7}{4}\right)$$

$X(f)$ is again shown on the right at the top.
Using $\tau\,\mathrm{sinc}(\tau t) \overset{\mathcal{F}}{\leftrightarrow} \Pi\left(\frac{f}{\tau}\right)$, (6.13), in combination with the modulation theorem, (6.7), $x_1(t)\cos(2\pi f_0 t) \overset{\mathcal{F}}{\leftrightarrow} \frac{1}{2}X_1(f+f_0)+\frac{1}{2}X_1(f-f_0)$, we obtain for Fourier transform $Y(f)$ of signal $y(t) = 4\,\mathrm{sinc}(4t)\cos(14\pi t)$:

$$Y(f) = \tfrac{1}{2}\Pi\left(\frac{f+7}{4}\right) + \tfrac{1}{2}\Pi\left(\frac{f-7}{4}\right)$$

$Y(f)$ is shown on the right at the bottom.



Clearly, when looking at the illustrations of $X(f)$ and $Y(f)$, output $Y(f)$ can be obtained by filtering $X(f)$ with an ideal bandpass filter with frequency response function $H_{BP}(f)$ with an amplitude, or so-called gain, of $\frac{4}{3}$, and that maintains the part of input signal $x(t)$ for absolute frequencies from $|f| = 5$ to $|f| = 9$ Hz, and that filters out the part of the signal for absolute frequencies from $f = 0$ to $|f| = 2$ Hz. When setting bandwidth frequencies of $f_1 = 4$ Hz and $f_2 = 10$ Hz, we achieve this result. Hence, $H_{BP}(f) = \frac{4}{3}\Pi\left(\frac{f+7}{6}\right) + \frac{4}{3}\Pi\left(\frac{f-7}{6}\right)$ as illustrated in the second row on the right, would work. Alternatively, an ideal high-pass filter with frequency response function $H_{HP}(f)$ with the same gain (of $\frac{4}{3}$) that filters out the part of the signal for absolute frequencies from $f = 0$ to $|f| = 2$ Hz, for example, by setting a bandwidth frequency $f_1 = 4$ Hz, would give the same result, as illustrated in the third row on the right. Note that, similar to (a), the bandwidth frequencies of the ideal filters have been set rather arbitrarily. As long as (1) the filter gain is set to $\frac{4}{3}$, (2) the filter lets input signal components pass for absolute frequencies from $|f| = 5$ to $|f| = 9$ Hz, and (3) the filter blocks input signal components for absolute frequencies from $f = 0$ to $|f| = 2$ Hz, output signal $y(t)$, resulting from filtering specified input signal $x(t)$, will be as requested.

S.18-3

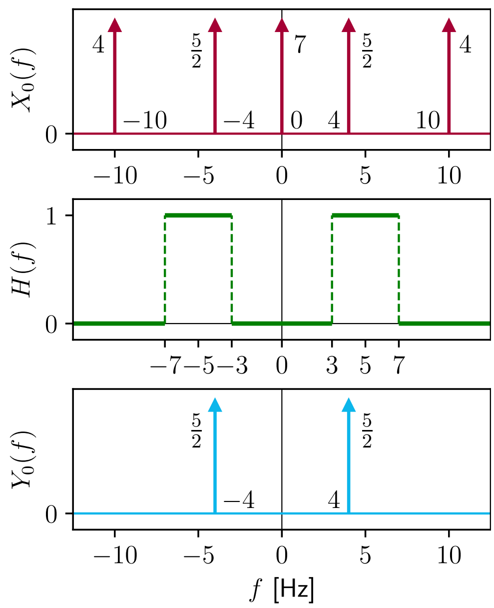
(a) Using $\sin^2 u = \frac{1}{2}(1 - \cos(2u))$, (A.5), to rewrite $x(t)$:

$$x(t) = 2 + 10\sin^2(4\pi t) + 8\cos\left(20\pi t + \frac{\pi}{12}\right) = 2 + 5 - 5\cos(8\pi t) + 8\cos\left(20\pi t + \frac{\pi}{12}\right)$$
$$= 7 - 5\cos(8\pi t) + 8\cos\left(20\pi t + \frac{\pi}{12}\right) = 7 + 5\cos(8\pi t + \pi) + 8\cos\left(20\pi t + \frac{\pi}{12}\right)$$

As only power calculations are required, the non-zero phases can be ignored, as signal power is independent of signal phase. Hence, in the remainder of this solution, we consider signal (with index 0): $x_0(t) = 7 + 5\cos(8\pi t) + 8\cos(20\pi t)$

Using $1 \overset{\mathcal{F}}{\leftrightarrow} \delta(f)$, (5.15), and $\cos(2\pi f_0 t) \overset{\mathcal{F}}{\leftrightarrow} \frac{1}{2}(\delta(f+f_0) + \delta(f-f_0))$, (5.16), we obtain for Fourier transform $X_0(f)$ of signal $x_0(t)$:

$$X_0(f) = 7\delta(f) + \frac{5}{2}(\delta(f+4) + \delta(f-4)) + 4(\delta(f+10) + \delta(f-10))$$



$X_0(f)$ is shown on the right at the top, with frequency response function $H(f)$ of the ideal bandpass filter underneath. Through $Y_0(f) = H(f)X_0(f)$, (16.7), we find that:

$$Y_0(f) = \frac{5}{2}(\delta(f+4) + \delta(f-4))$$

$Y_0(f)$ shown on the right at the bottom. The output signal power can then be computed by squaring $Y(f)$, or $Y_0(f)$ for that sake, and summing (integrating) the power of all components:

$$P_y = \left(\frac{5}{2}\right)^2 + \left(\frac{5}{2}\right)^2 = 12.5 \text{ W}$$

(b) Using $\cos^2 u = \frac{1}{2}(1 + \cos(2u))$, (A.4), to rewrite $x(t)$:

$$x(t) = 10\cos^2\left(20\pi t + \frac{\pi}{3}\right) - 5\sin\left(100\pi t + \frac{\pi}{2}\right)$$
$$= 10 \cdot \frac{1}{2}\left(1 + \cos\left(40\pi t + \frac{2\pi}{3}\right)\right) - 5\cos(100\pi t)$$
$$= 5 + 5\cos\left(40\pi t + \frac{2\pi}{3}\right) + 5\cos(100\pi t + \pi)$$

Similar to (a), as only power calculations are required, the non-zero phases can be ignored, as signal power is independent of signal phase. Hence, in the remainder of this solution, we consider signal (with index 0):

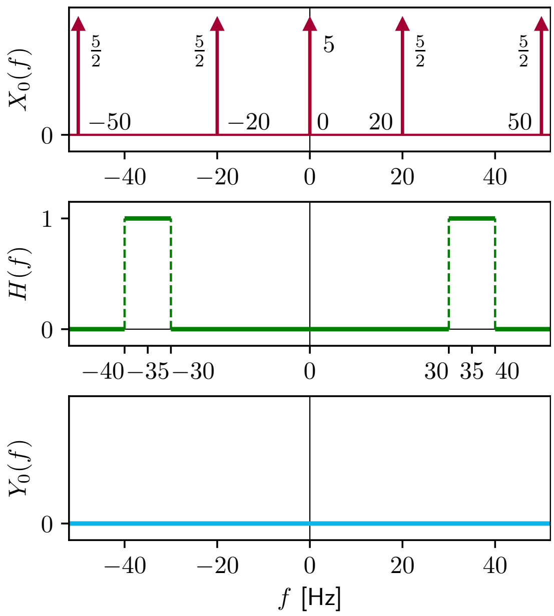
$$x_0(t) = 5 + 5\cos(40\pi t) + 5\cos(100\pi t)$$

Using $1 \overset{\mathcal{F}}{\leftrightarrow} \delta(f)$, (5.15), and $\cos(2\pi f_0 t) \overset{\mathcal{F}}{\leftrightarrow} \frac{1}{2}\big(\delta(f+f_0)+\delta(f-f_0)\big)$, (5.16), we obtain for Fourier transform $X_0(f)$ of signal $x_0(t)$:

$$\begin{aligned} X_0(f) &= 5\delta(f) \\ &\quad + \tfrac{5}{2}\big(\delta(f+20)+\delta(f-20)\big) \\ &\quad + \tfrac{5}{2}\big(\delta(f+50)+\delta(f-50)\big) \end{aligned}$$

$X_0(f)$ is shown on the right at the top, with frequency response function $H(f)$ of the ideal bandpass filter underneath. As can be seen, the input does not have any non-zero frequency components within the pass band of the filter. Hence, the power of output signal $y(t)$ equals:

$$P_y = 0 \text{ W}$$



S.18-4

(a) We obtain $x(t)$ through inverse Fourier-transforming $X(f) = \cos(\pi f)\,\Pi(f) = A(f)B(f)$ with $A(f) = \cos(\pi f)$ and $B(f) = \Pi(f)$. Using $\tau\,\mathrm{sinc}(\tau t) \overset{\mathcal{F}}{\leftrightarrow} \Pi\big(\frac{f}{\tau}\big)$, (6.13), and $\cos(2\pi f_0 t) \overset{\mathcal{F}}{\leftrightarrow} \frac{1}{2}\big(\delta(f+f_0)+\delta(f-f_0)\big)$, (5.16), in combination with the duality theorem, $X(t) \overset{\mathcal{F}}{\leftrightarrow} x(-f)$, (6.5), and $f_0 = \frac{1}{2}$, we obtain for time-domain signals $a(t)$ and $b(t)$:

$$a(t) = \tfrac{1}{2}\Big(\delta\big(t+\tfrac{1}{2}\big)+\delta\big(t-\tfrac{1}{2}\big)\Big) \qquad \text{and} \qquad b(t) = \mathrm{sinc}(t)$$

A multiplication in the frequency domain is equivalent to a convolution in the time domain, (6.8). Convolving a signal with the Dirac pulse $\delta(t-t_0)$ creates a copy of that signal, where the signal value at $t=0$ gets positioned at the Dirac pulse position $t=t_0$, (7.4):

$$x(t) = a(t) * b(t) = \tfrac{1}{2}\Big(\mathrm{sinc}\big(t+\tfrac{1}{2}\big)+\mathrm{sinc}\big(t-\tfrac{1}{2}\big)\Big)$$

You could check this result through Fourier-transforming $x(t)$ and see if you obtain $X(f)$.

(b) $Y(f) = 2X(f+f_0) + 2X(f-f_0) = 2\big(X(f+f_0)+X(f-f_0)\big)$ means that we have two copies of $X(f)$, one at $f=-f_0$ and one at $f=f_0$, scaled by a factor 2. In the frequency domain, we obtain these two copies of $X(f)$ when we convolve $X(f)$ with Dirac pulses at $f=-f_0$ and $f=f_0$. Hence:

$$Y(f) = 2\Big(X(f) * \big(\delta(f+f_0)+\delta(f-f_0)\big)\Big)$$

A convolution in the frequency domain is equivalent to a multiplication in the time domain, (6.9). In combination with $\cos(2\pi f_0 t) \overset{\mathcal{F}}{\leftrightarrow} \frac{1}{2}\big(\delta(f+f_0)+\delta(f-f_0)\big)$, (5.16), and the linearity theorem (6.1), we obtain:

$$\begin{aligned} y(t) &= 2\left(x(t)\,\mathcal{F}^{-1}\Big\{2\cdot\tfrac{1}{2}\big(\delta(f+f_0)+\delta(f-f_0)\big)\Big\}\right) = 2\,\big(x(t)\,2\cos(2\pi f_0 t)\big) \\ &= 4\,x(t)\cos(2\pi f_0 t) \end{aligned}$$

With $x(t) = \frac{1}{2}\Big(\mathrm{sinc}\big(t+\frac{1}{2}\big)+\mathrm{sinc}\big(t-\frac{1}{2}\big)\Big)$, we obtain:

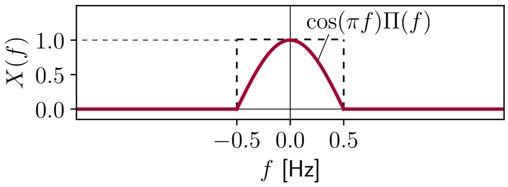
$$y(t) = 2\Big(\mathrm{sinc}\big(t+\tfrac{1}{2}\big)+\mathrm{sinc}\big(t-\tfrac{1}{2}\big)\Big)\cos(2\pi f_0 t)$$

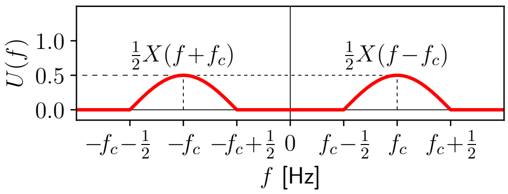
(c) The exercise states that we first multiply $x(t)$ by $\cos(2\pi f_c t)$ to obtain signal $u(t)$, and then feed it through an ideal low-pass filter with frequency response function $H(f)$, with gain G and bandwidth frequency f_2, to obtain signal $y(t)$.

We know that $X(f) = \cos(\pi f)\,\Pi(f)$, which is illustrated on the right. Given signal $u(t) = x(t)\cos(2\pi f_c t)$, the modulation theorem, $x(t)\cos(2\pi f_0 t) \overset{\mathcal{F}}{\leftrightarrow} \frac{1}{2}X(f+f_0) + \frac{1}{2}X(f-f_0)$, (6.7), states that, in this case, one copy of the Fourier transform of $x(t)$, $X(f)$, is shifted by $-f_c$ and another one by f_c, and both are multiplied by $\frac{1}{2}$. Hence:

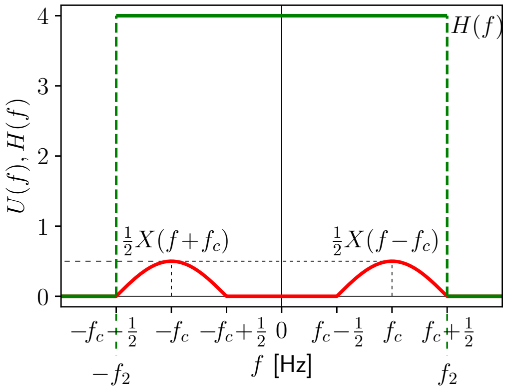
$$U(f) = \tfrac{1}{2}X(f+f_c) + \tfrac{1}{2}X(f-f_c)$$

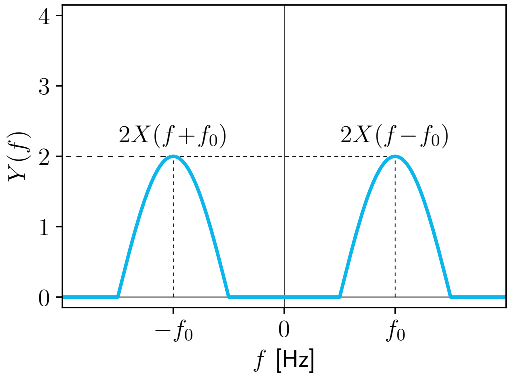
which is illustrated on the right.





We also know from the exercise description that $Y(f) = 2X(f+f_0) + 2X(f-f_0)$. Clearly, this can only be the case when f_0 equals f_c (that is, the frequency of the cosine that is multiplied by $x(t)$ equals $f_c = f_0$), and when $G = 4$ (that is, the gain of the ideal low-pass filter equals 4, as $4 \cdot \frac{1}{2} = 2$). Also, f_2 must be chosen such that the entire spectrum is passed by the filter, i.e., $f_2 > f_c + \frac{1}{2}$. See the illustration below on the left. Indeed, multiplying $H(f)$ by $U(f)$ yields $Y(f)$, shown below on the right.





Concluding: $G = 4$, $f_c = f_0$ and $f_2 > f_c + \frac{1}{2}$ will yield the correct result.

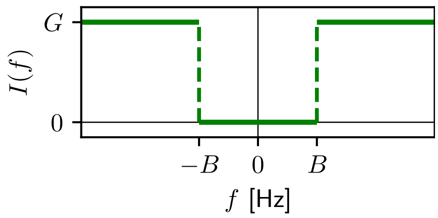
S.18-5

Impulse response function $h_{HP}(t)$ follows from inverse Fourier-transforming the frequency response function $H_{HP}(f) = G\left(1 - \Pi\left(\frac{f}{2B}\right)\right)e^{-j2\pi f t_0}$. $H_{HP}(f)$ consists of a multiplication of:

(I) $G\left(1 - \Pi\left(\frac{f}{2B}\right)\right)$ and (II) $e^{-j2\pi f t_0}$

We recognize part (II) to be something that is probably caused by a time delay, as we know from the time shift Fourier transform theorem that if we have $x(t) \overset{\mathcal{F}}{\leftrightarrow} X(f)$, then $x(t-t_0) \overset{\mathcal{F}}{\leftrightarrow} X(f)e^{-j2\pi f t_0}$, (6.2).

Now let us sketch part (I), here on the right, and call it $I(f)$. We see that, indeed, all lower frequencies are blocked, and the higher frequencies are passed. Inverse Fourier-transforming $I(f)$ is done as follows:



$$i(t) = \mathcal{F}^{-1}\left\{G\left(1 - \Pi\left(\frac{f}{2B}\right)\right)\right\} = G\mathcal{F}^{-1}\{1\} - G\mathcal{F}^{-1}\left\{\Pi\left(\frac{f}{2B}\right)\right\}$$

Using $\Pi\left(\frac{t}{\tau}\right) \overset{\mathcal{F}}{\leftrightarrow} \tau\,\mathrm{sinc}(\tau f)$, (5.11), and the duality theorem, $X(t) \overset{\mathcal{F}}{\leftrightarrow} x(-f)$, (6.5), we obtain for the second inverse Fourier transform component:

$$\mathcal{F}^{-1}\left\{\Pi\left(\frac{f}{2B}\right)\right\} = 2B\,\mathrm{sinc}(-t2B) = 2B\,\mathrm{sinc}(2Bt)$$

where in the latter step we made use of the fact that the sinc function is even.
Then, using $\delta(t) \overset{\mathcal{F}}{\leftrightarrow} 1$, (5.14), we obtain for the inverse Fourier transform of $I(f)$, or part (I):

$$i(t) = G\big(\delta(t) - 2B\,\mathrm{sinc}(2Bt)\big)$$

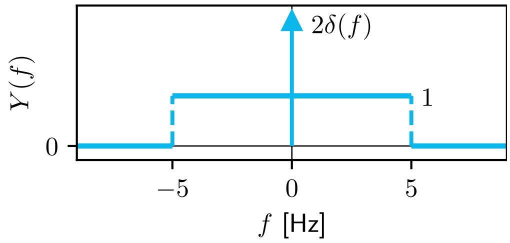
When including part (II), we must conclude that $h_{HP}(t)$ is a time-delayed version of $i(t)$:

$$h_{HP}(t) = i(t-t_0) = G\Big(\delta(t-t_0) - 2B\,\mathrm{sinc}\big(2B(t-t_0)\big)\Big)$$

S.18-6 The correct answer is B.

From the illustration, it can be deduced that the frequency response function of the low-pass filter is defined as:

$$H(f) = 2\Pi\left(\frac{f}{10}\right)$$



From $1 \overset{\mathcal{F}}{\leftrightarrow} \delta(f)$, (5.15), and $\tau\,\mathrm{sinc}(\tau t) \overset{\mathcal{F}}{\leftrightarrow} \Pi\left(\frac{f}{\tau}\right)$, (6.13), and using the linearity theorem (6.1), we obtain for the Fourier transform of output signal $y(t) = 2 + 10\,\mathrm{sinc}(10t)$:

$$Y(f) = 2\delta(f) + \Pi\left(\frac{f}{10}\right)$$

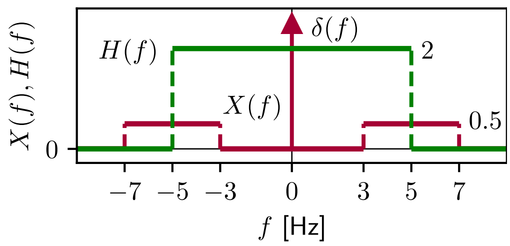
$Y(f)$ is illustrated above on the right in blue, where the Dirac pulse cannot be drawn to scale, as it resembles an extremely narrow, extremely high spike with an area, or weight, of 2.

We know that in the frequency domain the filtering operation can be described by $Y(f) = X(f)H(f)$, (16.7). Now let us have a look at whether any of the given options for $x(t)$ has a Fourier transform $X(f)$ that, when multiplied by $H(f)$, results in the above obtained $Y(f)$.

Option A. $x(t) = 1 + 4\,\mathrm{sinc}(4t)\cos(10\pi t)$:

Using the above mentioned Fourier transform pairs and linearity theorem, in combination with the modulation theorem, $x_1(t)\cos(2\pi f_0 t) \overset{\mathcal{F}}{\leftrightarrow} \frac{1}{2}X_1(f+f_0) + \frac{1}{2}X_1(f-f_0)$, (6.7), we obtain:

$$X(f) = \delta(f) + \tfrac{1}{2}\Pi\left(\frac{f+5}{4}\right) + \tfrac{1}{2}\Pi\left(\frac{f-5}{4}\right)$$

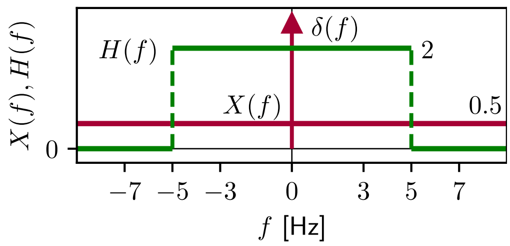


$X(f)$ is illustrated on the right in burgundy, together with $H(f)$ in green. Without having to exactly determine $X(f)H(f)$, we can see that this $X(f)$ does not result in the above described and illustrated $Y(f)$ when multiplied by $H(f)$.
Hence, answer A. is incorrect.

Option B. $x(t) = 1 + 10\,\mathrm{sinc}(10t)\cos(10\pi t)$:

Using the same Fourier transform pairs and theorems, we obtain:

$$X(f) = \delta(f) + \tfrac{1}{2}\Pi\left(\frac{f+5}{10}\right) + \tfrac{1}{2}\Pi\left(\frac{f-5}{10}\right)$$



$X(f)$ is illustrated on the right in burgundy, together with $H(f)$ in green. It is difficult to recognize the two pulse functions, as one is centered at $f = -5$ Hz, with a width of 10 Hz, and one is centered at $f = 5$ Hz, with an equal width of 10 Hz, resulting in a single pulse function centered at 0 Hz with a width of 20 Hz, i.e., from $f = -10$ to $f = 10$ Hz. Remember that at the edge of a pulse function, the function value is defined to be half of the pulse's amplitude, (B.1). Hence, in this case, at $f = 0$, at which the two pulse functions 'meet', the function value at the right edge of the left pulse function and the value at the left edge of the right pulse function are summed, resulting in a value equal to the pulses' amplitude. We thus obtain a single pulse function centered at 0 Hz with a width of 20 Hz. When multiplying $X(f)$ by $H(f)$, with bandwidth frequency $f_2 = 5$ Hz and a gain equal to 2, one can see that this does indeed result in:

$$Y(f) = 2\delta(f) + \Pi\left(\frac{f}{10}\right)$$

Hence, answer B. is correct, from which it follows that answer D. is incorrect.

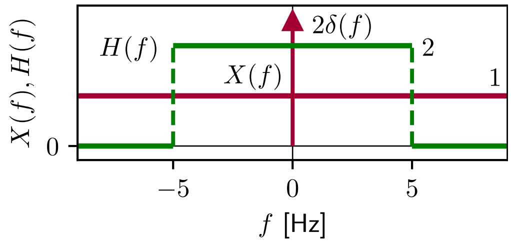
As a final check, below we confirm that answer C. is incorrect.

Option C. $\quad x(t) = 2 + 20\,\mathrm{sinc}(20t)$:

Using the same Fourier transform pairs and theorems again, we obtain:

$$X(f) = 2\delta(f) + \Pi\left(\frac{f}{20}\right)$$

For the sake of completeness, $X(f)$ is illustrated on the right in burgundy, together with $H(f)$ in green. However, without having to sketch $X(f)$, we know that when multiplying $X(f)$ by $H(f)$, which has a gain of 2, the result will not be equal to the above described $Y(f)$.



s | 18.2 Practical filters

S.18-7

(a) Given $u_{\text{out}}(t) = R\,i(t)$, with $i(t) = C\frac{\mathrm{d}}{\mathrm{d}t}\big(u_{\text{in}}(t) - u_{\text{out}}(t)\big)$, we obtain:

$$u_{\text{out}}(t) = RC\frac{\mathrm{d}}{\mathrm{d}t}\big(u_{\text{in}}(t) - u_{\text{out}}(t)\big) = RC\frac{\mathrm{d}}{\mathrm{d}t}u_{\text{in}}(t) - RC\frac{\mathrm{d}}{\mathrm{d}t}u_{\text{out}}(t)$$

Rearranging yields:

$$RC\frac{\mathrm{d}}{\mathrm{d}t}u_{\text{out}}(t) + u_{\text{out}}(t) = RC\frac{\mathrm{d}}{\mathrm{d}t}u_{\text{in}}(t)$$

With $u_{\text{in}}(t) = x(t)$ and $u_{\text{out}}(t) = y(t)$, we obtain:

$$RC\frac{\mathrm{d}y(t)}{\mathrm{d}t} + y(t) = RC\frac{\mathrm{d}x(t)}{\mathrm{d}t}$$

(b) Fourier-transforming the ordinary differential equation obtained in (a) using the differentiation Fourier transform theorem (6.10):

$$RC(j2\pi f)Y(f) + Y(f) = RC(j2\pi f)X(f)$$

which can be rewritten as:

$$\big(RC(j2\pi f) + 1\big)Y(f) = RC(j2\pi f)X(f)$$

Hence, using (16.7), we obtain:

$$H(f) = \frac{Y(f)}{X(f)} = \frac{j2\pi fRC}{1 + j2\pi fRC} \qquad \text{or with } f_3 = \frac{1}{2\pi RC}\text{:} \qquad H(f) = \frac{j\frac{f}{f_3}}{1 + j\frac{f}{f_3}}$$

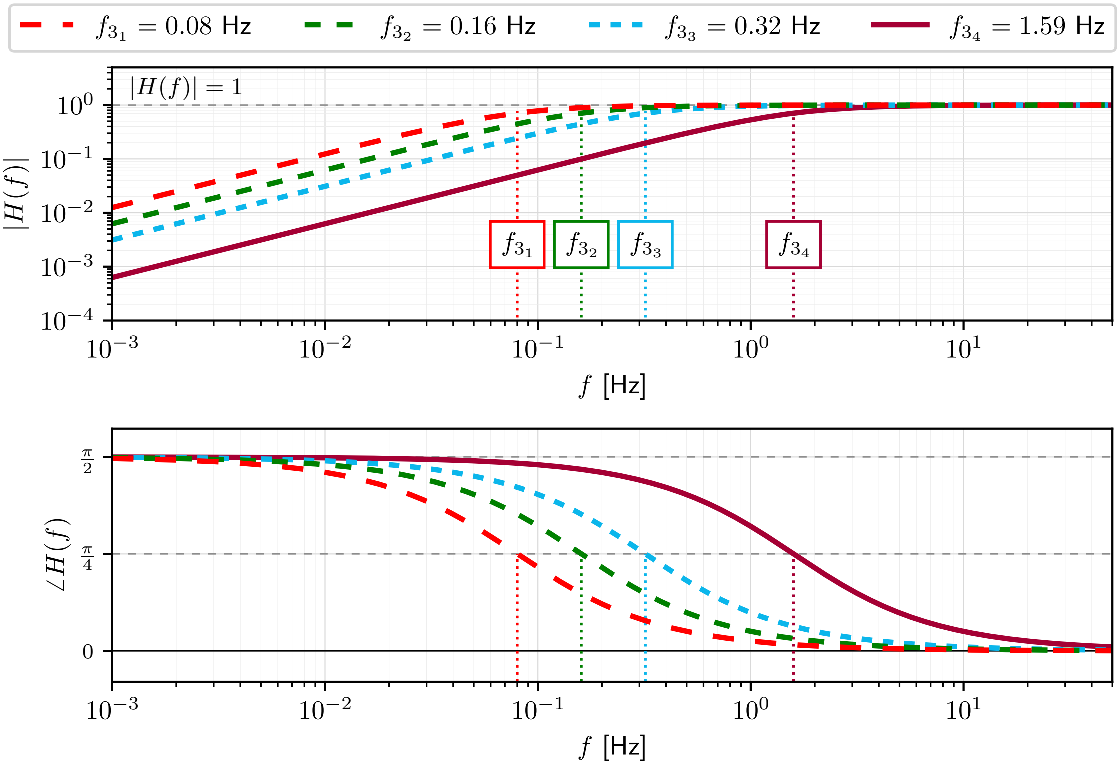
(c) Frequency f_3 can be seen as the 'bandwidth' of the RC high-pass filter. The figure at the bottom of this solution shows a Bode plot of $H(f)$ obtained in (b) for various values of f_3. From the Bode plot, we can see that $H(f)$ equals a differentiator when $f \ll f_3$.

However, creating a Bode plot of frequency response function $H(f)$ was not necessarily required to answer this question. We could also have concluded that $H(f)$ equals a differentiator when $f \ll f_3$ from the frequency response function obtained in (b):

$$H(f) = \frac{j\frac{f}{f_3}}{1 + j\frac{f}{f_3}}$$

Namely, when $f \ll f_3$, then $1 \gg \frac{f}{f_3}$ in the denominator, and $H(f) \approx j\frac{f}{f_3}$, i.e., $H(f) \approx (j2\pi f)RC$, which equals differentiator dynamics, cf. (6.10).

In other words, for signals that have frequencies in the region $0 < f \ll f_3$, the RC high-pass filter will act as a differentiator. For frequencies in the region $f \gg f_3$, the filter simply 'passes' the signal content, hence, the name 'high-pass filter'.



s | 18.c Computer exercises

s.18-8 The script to create the Bode plot of (a), is included below.

Basic Python code:

```
import numpy as np
from matplotlib import pyplot as plt

burgundy = [165/255,0/255,52/255]            # burgundy color

# Bode plot of frequency response function H(f)
# of resistor-capacitor (RC) high-pass filter
# (only the code for a single value of f_3 is shown)

# frequency vector and f_3
NN  = 50000; f = np.linspace(0.001, 50, NN);    # frequency vector
f34 = 1.59                                      # frequency f_3 [Hz]
```

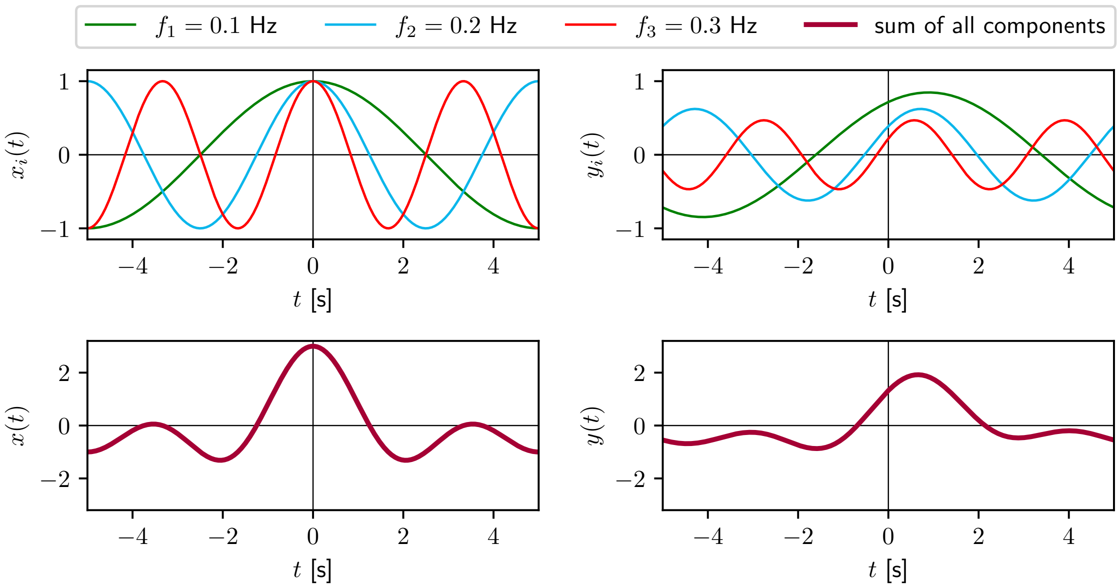
```
# frequency response function H(f)
df34  = 1 + (f / f34)**2                          # denominator
Rf34  = ((f / f34)**2) / df34                     # real part
If34  = ((f / f34)) / df34                        # imaginary part
mHf34 = np.sqrt(Rf34**2 + If34**2)                # magnitude response
pHf34 = np.arctan2(If34, Rf34)                    # phase response

# Bode plot of frequency response function H(f)
plt.figure()
plt.subplot(3,1,1)                                # magnitude response
plt.loglog(f, mHf34, color=burgundy,linewidth=2,label='$f_{34}=1.59$ Hz')
plt.xlabel('$f$ [Hz]'); plt.ylabel('$|H(f)|$')

plt.subplot(3,1,2)                                # phase response
plt.semilogx(f, pHf34, color=burgundy,linewidth=2)
plt.xlabel('$f$ [Hz]'); plt.ylabel('$\\angle H(f)$')
```

S.18-9

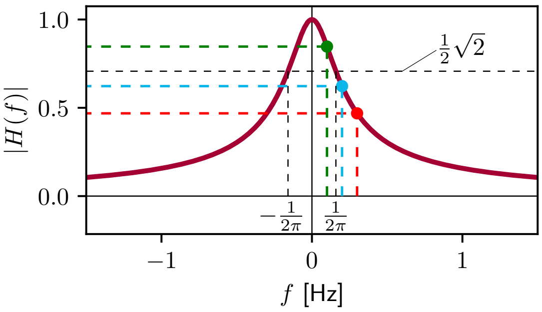
(a)-(b) Section 16.4 shows the output of a linear time-invariant system to a harmonic input.
An input signal $x(t) = A\cos(2\pi f_0 t)$ (16.12) yields
output signal $y(t) = A|H(f_0)|\cos\big(2\pi f_0 t + \theta(f_0)\big)$ (16.14) with $\theta(f_0) = \angle H(f_0)$
For input signal $x(t) = \cos(2\pi f_1 t) + \cos(2\pi f_2 t) + \cos(2\pi f_3 t)$, when using the linearity theorem (6.1), we arrive at:

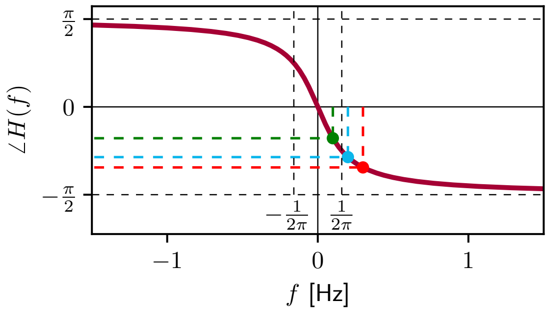
$$y(t) = |H(f_1)|\cos\big(2\pi f_1 t + \theta(f_1)\big) + |H(f_2)|\cos\big(2\pi f_2 t + \theta(f_2)\big) + |H(f_3)|\cos\big(2\pi f_3 t + \theta(f_3)\big)$$

The input (at left) and output (at right) signals are shown below. The top row shows the individual components in green ($f_1 = 0.1$ Hz), blue ($f_2 = 0.2$ Hz) and red ($f_3 = 0.3$ Hz). The bottom row shows the full input signal $x(t)$ at left and output signal $y(t)$ at right.



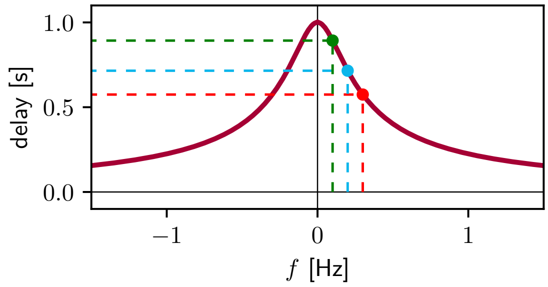
(c) The figure below, which is a reproduction of Figure 18.8, presents the magnitude response $|H(f)|$ and phase response $\angle H(f)$ of the first-order Butterworth filter, with the frequencies of the three individual cosines explicitly indicated in their respective colors.





The attenuation, $|H(f_0)| < 1$, is listed in the table below at left. The attenuation is clearly different for each of the three cosines, as seen in the top right plot of the figure in (a)-(b). The phase change is also different for each of the three cosines. According to Section 16.4.2, a phase change $\angle H(f_0) = \theta(f_0)$ (in [rad]) manifests itself as a time shift $t_0 = \frac{\theta(f_0)}{2\pi f_0}$. For each cosine component, the output has a negative time shift (see the figure above at right and the table below at left). This means that the output lags behind the input, a time delay (see the figure below at right, a reproduction of Figure 18.10 at left). As can be seen from the figure above at right, the first-order Butterworth filter clearly has a nonlinear phase response. The result is that the signal gets distorted or deformed; the filter changes the shape of the signal.

	f_i [Hz]	$\lvert H(f_0)\rvert$	$\angle H(f_0)$	t_0 [s]
1	0.1	0.85	−0.56	−0.89
2	0.2	0.62	−0.90	−0.72
3	0.3	0.47	−1.08	−0.57



Basic Python code:

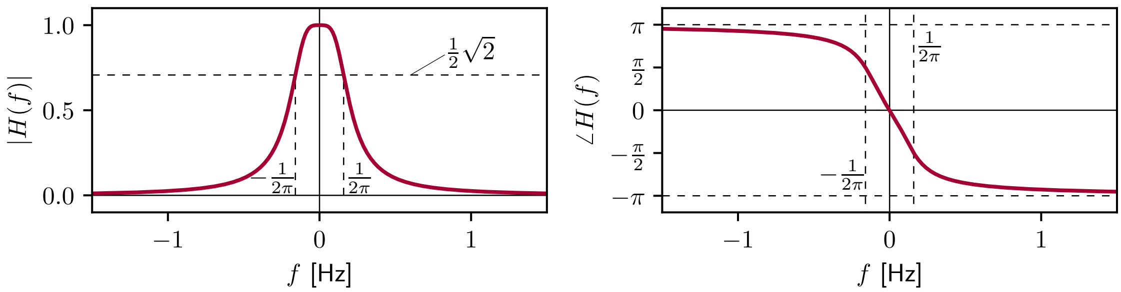
```
import numpy as np
from matplotlib import pyplot as plt

burgundy = [165/255,0/255,52/255]      # burgundy color

# demonstrate effect of filtering a signal (in continuous time),
# where signal consists of three cosines (at different frequencies)
# frequency response function of first-order Butterworth low-pass filter
# amplitude |H(f)| and phase arg(H(f)) depend on frequency
# output signal turns out to be distorted compared to input signal

# continuous time approximated by Delta_t = 0.01 s
N = 1001                               # number of time/frequency steps
t = np.linspace(-5.0, 5.0, N)          # time steps 0.01 s from -5 s to 5 s

# constants of three cosines
f1 = 0.1; f2 = 0.2; f3 = 0.3           # cosine frequency [Hz]
A1 = 1;    A2 = 1;    A3 = 1           # amplitude

# create multi-cosine input signal x(t)
x1 = A1 * np.cos(2*np.pi*f1*t)
x2 = A2 * np.cos(2*np.pi*f2*t)
```

```
x3 = A3 * np.cos(2*np.pi*f3*t)
x  = x1 + x2 + x3

# frequency response function evaluated at the three frequencies
Hf1 = 1 / ( 1 + 2j*np.pi*f1 )
Hf2 = 1 / ( 1 + 2j*np.pi*f2 )
Hf3 = 1 / ( 1 + 2j*np.pi*f3 )

# create output signal y(t)
y1 = A1 * np.abs(Hf1) * np.cos( 2*np.pi*f1*t + np.angle(Hf1) )
y2 = A2 * np.abs(Hf2) * np.cos( 2*np.pi*f2*t + np.angle(Hf2) )
y3 = A3 * np.abs(Hf3) * np.cos( 2*np.pi*f3*t + np.angle(Hf3) )
y  = y1 + y2 + y3

# plot of individual cosine components x_i(t)
plt.figure()
plt.subplot(2,2,1)
plt.plot(t, x1, color='g', linewidth=1)
plt.plot(t, x2, color='b', linewidth=1)
plt.plot(t, x3, color='r', linewidth=1)
plt.xlabel('$t$ [s]'); plt.ylabel('$x_{i}(t)$')

# plot of individual filtered components y_i(t)
plt.subplot(2,2,2)
plt.plot(t, y1, color='g', linewidth=1)
plt.plot(t, y2, color='b', linewidth=1)
plt.plot(t, y3, color='r', linewidth=1)
plt.xlabel('$t$ [s]'); plt.ylabel('$y_{i}(t)$')

# plot of full input signal x(t)
plt.subplot(2,2,3)
plt.plot(t, x, color=burgundy, linewidth=2)
plt.xlabel('$t$ [s]'); plt.ylabel('$x(t)$')

# plot of full output signal y(t)
plt.subplot(2,2,4)
plt.plot(t, y, color=burgundy, linewidth=2)
plt.xlabel('$t$ [s]'); plt.ylabel('$y(t)$')

# plot frequency response function of first-order Butterworth LP filter
# amplitude |H(f)| and phase arg(H(f)) as function of linear frequency f

# continuous frequency approximated by Delta_f = 0.01 Hz
f = np.linspace(-5.0, 5.0, N) # frequency steps 0.01 Hz from -5 Hz to 5 Hz
Hn = np.ones(N)               # numerator
Hd = np.ones(N) + 2j*np.pi*f  # denominator
Hf = np.divide(Hn, Hd)        # frequency response function

plt.figure()
plt.subplot(3,2,1)            # amplitude spectrum
plt.plot(f,  np.abs(Hf),  color=burgundy, linewidth=2)
plt.plot(f1, np.abs(Hf1), color='g',  marker='o', markersize=4)
plt.plot(f2, np.abs(Hf2), color='b',  marker='o', markersize=4)
plt.plot(f3, np.abs(Hf3), color='r',  marker='o', markersize=4)
plt.xlabel('$f$ [Hz]'); plt.ylabel('$|H(f)|$')
```

```
plt.subplot(3,2,2)              # phase spectrum
plt.plot(f,  np.angle(Hf),  color=burgundy, linewidth=2)
plt.plot(f1, np.angle(Hf1), color='g',  marker='o', markersize=4)
plt.plot(f2, np.angle(Hf2), color='b',  marker='o', markersize=4)
plt.plot(f3, np.angle(Hf3), color='r',  marker='o', markersize=4)
plt.xlabel('$f$ [Hz]'); plt.ylabel('$\\angle H(f)$')

# plot time delay of first-order Butterworth low-pass filter

Hftimedelay  = np.divide(np.arctan(2*np.pi*f),  2*np.pi*f)
Hftimedelay1 = np.divide(np.arctan(2*np.pi*f1), 2*np.pi*f1)
Hftimedelay2 = np.divide(np.arctan(2*np.pi*f2), 2*np.pi*f2)
Hftimedelay3 = np.divide(np.arctan(2*np.pi*f3), 2*np.pi*f3)

plt.subplot(3,2,3)
plt.plot(f,  Hftimedelay,  color=burgundy, linewidth=2)
plt.plot(f1, Hftimedelay1, color='g', marker='o', markersize=4)
plt.plot(f2, Hftimedelay2, color='b', marker='o', markersize=4)
plt.plot(f3, Hftimedelay3, color='r', marker='o', markersize=4)
plt.xlabel('$f$ [Hz]'); plt.ylabel('delay [s]')
```

S.18-10

(a) The figure below shows magnitude response $|H(f)|$ (at left) and phase response $\angle H(f)$ (at right) of a second-order Butterworth low-pass filter with cut-off frequency $f_3 = \frac{1}{2\pi}$ Hz.



(b) Compared to a first-order Butterworth low-pass filter, a second-order Butterworth low-pass filter shows a faster (steeper) transition from pass band to stop band. Also higher-order Butterworth low-pass filters exist.

Basic Python code:

```
import numpy as np
from matplotlib import pyplot as plt

burgundy = [165/255,0/255,52/255] # burgundy color

# create frequency vector and define cut-off frequency f_3
f0     = 0.001           # high frequency resolution for visualization [Hz]
flim   = 1.5
f_vec  = np.arange(-flim, flim, f0)# frequency steps f0 from -1.5 to 1.5 Hz
f_3    = 1 / (2*np.pi)             # cut-off frequency [Hz]

# define frequency response H(f) for 2nd-order Butterworth low-pass filter
H = 1 / ( 1 + 1j*np.sqrt(2)*(f_vec/f_3) - (f_vec/f_3)**2 )
```

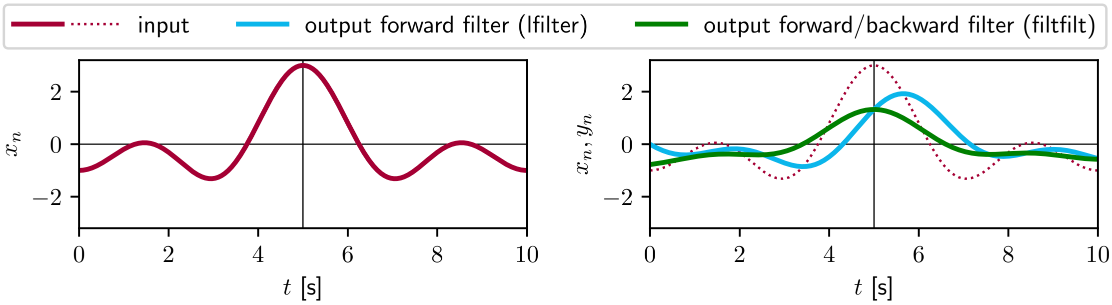
```
# compute absolute value and phase of H(f)
abs_H = np.abs(H); phase_H = np.angle(H)

# plot amplitude and phase spectra on linear scale
plt.figure()
plt.subplot(3,2,1)                                    # amplitude spectrum
plt.plot(f_vec, abs_H, color=burgundy, linewidth=1.5)
plt.xlabel('$f$ [Hz]'); plt.ylabel('$|H(f)|$')

plt.subplot(3,2,2)                                    # phase spectrum
plt.plot(f_vec, phase_H, color=burgundy, linewidth=1.5)
plt.xlabel('$f$ [Hz]'); plt.ylabel('$\\angle H(f)$')
```

s.18-11

(a)-(d) The figure below at left shows input sequence x_n in burgundy. It appears to be a continuous-time signal, as all samples are shown as a single continuous curve for clearer visualization, but it *is* a discrete-time signal, with $f_s = 100$ Hz. The figure below at right shows output sequence y_n, as a result of `scipy.signal.lfilter`, in blue. The signal gets deformed, and also the delay caused by the filter is clearly visible (as reference, the original signal x_n is shown as a burgundy dotted curve). The result is very similar to output signal $y(t)$ as constructed in continuous time with infinite length in Exercise *e*.18-9.

Differences between the blue curve y_n here and the burgundy curve $y(t)$ in the bottom right plot in Solution *s*.18-9(a)-(b), are noticeable primarily in the beginning, e.g., during the first 100 samples, due to the 'start-up' of the filter. It takes a while before the filter is fully engaged (the filter by default assumes the signal to start from zero; i.e., initial rest). We are working here with a *finite*-length signal.

In the figure below at right also output sequence y_n, as a result of `scipy.signal.filtfilt`, is shown in green. There is no delay, the signal is still aligned with the original signal shown by the burgundy dotted curve, and there is also hardly any deformation of the signal. The green curve is symmetric about $t = 5$ s, except for the (far) tails.



`scipy.signal.filtfilt` is a *zero-phase* filter, meaning that it does not cause any (frequency-dependent) time delay; the phase of the filter is zero at all frequencies. It does so by applying the filter twice! First forward, and then backward. `scipy.signal.lfilter` is a so-called forward filter, it works on the input sequence in one way; it is a causal filter, which can be applied in real time. `scipy.signal.filtfilt` can be used only in post-processing, as first the entire input data record needs to be available, in order to apply forward and backward filtering. With `scipy.signal.filtfilt` the signal amplitude (in green) is smaller than with `scipy.signal.lfilter` (in blue), as with the former, the filter has been applied twice!

Basic Python code:

```
import numpy as np
from scipy import signal
from matplotlib import pyplot as plt

# demonstrate effect of filtering a signal (in discrete time)
# using Python's lfilter and filtfilt
# signal consists of three cosines (at different frequencies)
# we apply a first-order Butterworth low-pass filter

burgundy = [165/255,0/255,52/255]     # burgundy color

# time vector
dt = 0.01                             # time interval or time step size [s]
t0 = 0                                # start time [s]
t1 = 10                               # end time [s]
t  = np.arange(t0, t1, dt)            # time steps 0.01 s from 0 s to 10 s

# constants of three cosines
f1 = 0.1; f2 = 0.2; f3 = 0.3          # cosine frequency [Hz]
A1 = 1;   A2 = 1;   A3 = 1            # amplitude

# create multi-cosine input signal (discrete time)
# time shift (t-5) to match continuous-time filter exercise
x1 = A1 * np.cos(2*np.pi*f1*(t-5))
x2 = A2 * np.cos(2*np.pi*f2*(t-5))
x3 = A3 * np.cos(2*np.pi*f3*(t-5))
x  = x1 + x2 + x3

# design and apply digital first-order Butterworth low-pass filter
# -3dB frequency is f3 = 1/(2pi) Hz, sampling frequency fs = 100 Hz
f_s = 100                                    # sampling frequency [Hz]
b,a = signal.butter(1, 1/(2*np.pi), fs=f_s) # filter coefficients
y   = signal.lfilter(b, a, x)                # linear (forward) filter
yff = signal.filtfilt(b, a, x)               # forward and backward filter

# plot of input signal
plt.figure()
plt.subplot(3,2,1)
plt.plot(t, x, color=burgundy, linewidth=2)
plt.xlabel('$t$ [s]'); plt.ylabel('$x_n$')

# plot of output signal
plt.subplot(3,2,2)
plt.plot(t, x, color=burgundy, linestyle=':', linewidth=1, label='input')
plt.plot(t, y, color='b', linewidth=2, label='output forward filter')
plt.plot(t, yff, color='g', linewidth=2, label='output for/backward filt')
plt.xlabel('$t$ [s]'); plt.ylabel('$x_n,y_n$')
```

S.18-12

(a) In the time domain, the steps can be described as follows:
1. The forward filtering step is modeled as $y(t) = x(t) * h(t)$ cf. (16.6)
2. The output signal is time-reversed: $y(t) \to y(-t)$
3. The time-reversed output is convolved with $h(t)$ in the backward filtering step: $z(t) = y(-t) * h(t)$
4. The output is time-reversed to restore the original forward time: $z(t) \to v(t) = z(-t)$

The corresponding steps in the frequency domain are:
1. $Y(f) = X(f)H(f)$ for the forward filter cf. (6.8), (16.7)
2. $y(-t) \overset{\mathcal{F}}{\leftrightarrow} Y(-f)$ for the time reversal cf. (6.4)
3. $Z(f) = Y(-f)H(f)$ for the backward filter
4. $v(t) = z(-t) \overset{\mathcal{F}}{\leftrightarrow} V(f) = Z(-f)$ for the time reversal

For a real $h(t)$:

$$H(-f) = H^*(f) \quad \text{cf. (5.10)}$$

Hence:

$$V(f) = Z(-f) = Y(f)H(-f) = Y(f)H^*(f) = X(f)H(f)H^*(f) = X(f)|H(f)|^2$$

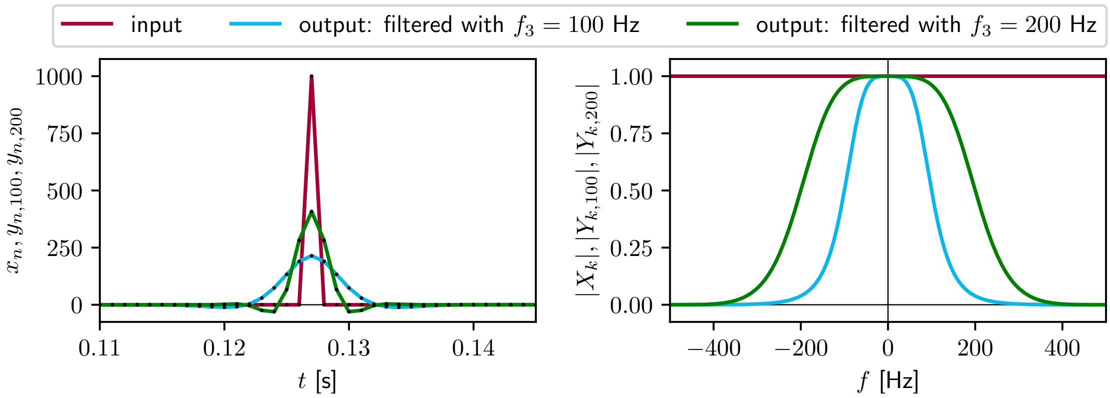
Concluding, output signal $v(t)$ of the double-filtering procedure has Fourier transform:

$$V(f) = X(f)|H(f)|^2$$

(b) Eventually Fourier transform $X(f)$ of input signal $x(t)$ is multiplied by the squared *magnitude* of the filter frequency response function $H(f)$, and this squared magnitude is just a *real* function of frequency f, leaving the phase of $X(f)$ unchanged.

S.18-13

(a)-(g) The figure below at left shows input sequence x_n (in burgundy) and output sequence y_n (filtered with: $f_3 = 100$ Hz in blue; $f_3 = 200$ Hz in green). Note that the discrete-time samples (black dots) are connected by line segments for clearer visualization, but the signals are discrete-time (not continuous-time) signals, with $f_s = 1000$ Hz. Only part of the 'restored' continuous time t is shown, such that the differences between the signals are clearly visible. The corresponding magnitude spectra are shown at right. As the discrete frequencies are so close to one another, the black dots have been left out, and only the colored line segments are shown.



(h) The filtered signals are more attenuated and have a wider pulse shape than the original signal. The Butterworth filter with cut-off frequency $f_3 = 100$ Hz allows *less* of the frequency content to pass through than the filter configured with cut-off frequency $f_3 = 200$ Hz. These characteristics can be seen in the magnitude spectra. The filtered signal in blue on the left is wider (and more attenuated) than the one in green.

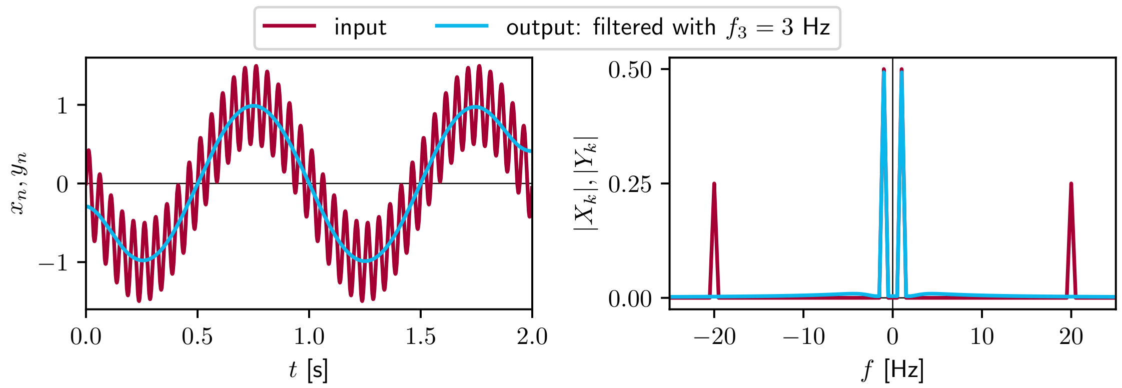
Basic Python code:

```
import numpy as np
from scipy import signal
from matplotlib import pyplot as plt

burgundy = [165/255,0/255,52/255]                # burgundy color

# time vector
dt       = 0.001                                 # time interval [s]
N        = 256                                   # number of samples
T_meas   = N * dt                                # total time [s]
time_sec = np.arange(0, T_meas, dt)              # time vector

# discrete-time sequence x_n
f_s    = 1 / dt      # sampling frequency [Hz]
x      = np.zeros(N) # signal vector with same length as time vector
x[127] = f_s         # Dirac pulse at sample number 128 (Python index 127)

# design and apply digital second-order Butterworth low-pass filter
f_3_100 = 100                                     # cut-off frequency [Hz]
b_100, a_100 = signal.butter(2, f_3_100, fs=f_s)  # filter coefficients
y_100 = signal.filtfilt(b_100, a_100, x)          # filtered signal

f_3_200 = 200                                     # cut-off frequency [Hz]
b_200, a_200 = signal.butter(2, f_3_200, fs=f_s)  # filter coefficients
y_200 = signal.filtfilt(b_200, a_200, x)          # filtered signal

# frequency vector
df    = 1 / T_meas                     # frequency step size [Hz]
f_vec = np.arange(-f_s/2, f_s/2, df)   # frequency axis from -500 to 500 Hz

# Discrete Fourier transforms of unfiltered and filtered signals
# multiply Python's FFT result by Delta_t to obtain DFT and
# shift the negative frequencies to the left-hand side such that
# the zero-frequency component appears in the center of the spectrum
Xk    = np.fft.fftshift(dt * np.fft.fft(x))
Yk100 = np.fft.fftshift(dt * np.fft.fft(y_100))
Yk200 = np.fft.fftshift(dt * np.fft.fft(y_200))

# plot unfiltered and filtered signals in time domain
plt.figure()
plt.subplot(2,2,1)
plt.plot(time_sec, x, color=burgundy, linewidth=1.5, label='original')
plt.plot(time_sec, y_100, color='b', linewidth=1.5, label='filtered 100')
plt.plot(time_sec, y_200, color='g', linewidth=1.5, label='filtered 200')
plt.plot(time_sec, x, '.', color='k', markersize=1)
plt.plot(time_sec, y_100, '.', color='k', markersize=1)
plt.plot(time_sec, y_200, '.', color='k', markersize=1)
plt.xlabel('$t$ [s]'); plt.ylabel('$x_n,y_{n,100},y_{n,200}$')

# plot amplitude spectra of unfiltered and filtered signals
plt.subplot(2,2,2)
plt.plot(f_vec, np.abs(Xk), color=burgundy, linewidth=1.5)
plt.plot(f_vec, np.abs(Yk100), color='b', linewidth=1.5)
```

```
plt.plot(f_vec, np.abs(Yk200), color='g', linewidth=1.5)
plt.xlabel('$f$ [Hz]'); plt.ylabel('$|X_k|,|Y_{k,100}|,|Y_{k,200}|$')
```

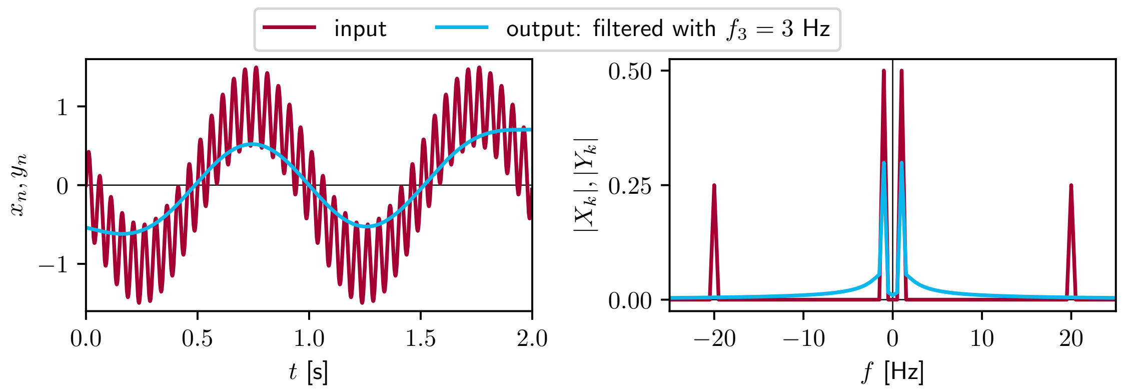
S.18-14

(a)-(e) The figure below at left shows input sequence x_n (in burgundy) and output sequence y_n (in blue; filtered with $f_3 = 3$ Hz). The used cut-off frequency $f_3 = 3$ Hz is in between the high frequency of 20 Hz and the low frequency of 1 Hz. Note that the discrete-time samples are shown as continuous curves because the samples are close to one another; however, they are *not* continuous-time signals. The corresponding magnitude spectra are shown at right. A limited frequency range is shown, to be able to clearly see the differences between the magnitude spectra of the unfiltered and filtered signals. However, in practice, the sequences X_k and Y_k cover the interval $[0, f_s)$, or $[-\frac{f_s}{2}, \frac{f_s}{2})$, the latter symmetric about $f = 0$, where $f_s = 1000$ Hz. Similar to the discrete-time sequences, as the discrete frequencies are so close to one another, the magnitude spectra are shown as continuous curves.



(f) In the figure above, a cut-off frequency of $f_3 = 3$ Hz was used for the second-order Butterworth low-pass filter. When f_3 is increased, the amplitude of the 1 Hz component in the filtered signal becomes slightly larger, i.e., even closer to that of the unfiltered signal (since the amplitude is slightly smaller in the filtered signal). When f_3 is increased further, the filtered signal appears less smooth in the time domain, because the higher-frequency sine is then attenuated less. Correspondingly, (small) peaks 'begin to reappear' in magnitude spectrum $|Y_k|$ at an absolute frequency of 20 Hz. When f_3 is increased too much, the higher-frequency sine is no longer attenuated.

When f_3 is decreased instead, not only is the 20 Hz sine attenuated, but the 1 Hz sine as well, which has a large influence on both the time-domain signal and its magnitude spectrum. The figure below shows the results for a second-order Butterworth low-pass filter with cut-off frequency $f_3 = 1$ Hz, rather than $f_3 = 3$ Hz as before.



Basic Python code:

```
import numpy as np
from scipy import signal
from matplotlib import pyplot as plt

burgundy = [165/255,0/255,52/255]     # burgundy color

# define required parameters
dt    = 0.001                         # time interval or time step size [s]
N     = 2000                          # number of samples
Tmeas = N * dt                        # total measurement time [s]
df    = 1 / Tmeas                     # frequency step size [Hz]
f_s   = 1 / dt                        # sampling frequency [Hz]

# create time vector and discrete-time sequence x_n
time_sec = np.arange(0, Tmeas, dt)    # time vector
A_1 = 0.5                             # amplitude sine 1
A_2 = 1                               # amplitude sine 2
f_1 = 20                              # frequency sine 1 [Hz]
f_2 = 1                               # frequency sine 2 [Hz]
xn  = A_1*np.sin(2*np.pi*f_1*time_sec) - A_2*np.sin(2*np.pi*f_2*time_sec)

# Discrete Fourier transform of unfiltered signal with amplitude scaling
# multiply Python's FFT result by Delta_t and divide by measurement time T
# to maintain equivalence with continuous-time signal (*dt/T)
f_vec = np.arange(-f_s/2, f_s/2, df) # frequency vector [-500 Hz, 500 Hz)
Xk    = np.fft.fft(xn) / N            # DFT;(*Delta_t/T equals divide by N)
Xk_shift = np.fft.fftshift(Xk)        # first negative, then positive freq

# design and apply digital second-order Butterworth low-pass filter
f_3      = 3                                # cut-off frequency [Hz]
b_3, a_3 = signal.butter(2, f_3, fs=f_s)    # filter coefficients
yn       = signal.filtfilt(b_3, a_3, xn)    # filtered signal

# Discrete Fourier transform of filtered signal
Yk       = np.fft.fft(yn) / N         # DFT;(*Delta_t/T equals divide by N)
Yk_shift = np.fft.fftshift(Yk)        # first negative, then positive freq

# plot unfiltered and filtered signals in time domain
plt.figure()
plt.subplot(3,2,1)
plt.plot(time_sec, xn, color=burgundy, label='input')
plt.plot(time_sec, yn, color='b', label='out: filtered with $f_3 = 3$ Hz')
plt.xlabel('$t$ [s]'); plt.ylabel('$x_n,y_n$')

# plot amplitude spectra of unfiltered and filtered signals
plt.subplot(3,2,2)
plt.plot(f_vec, np.abs(Xk_shift), color=burgundy, linewidth=1.5)
plt.plot(f_vec, np.abs(Yk_shift), color='b', linewidth=1.5)
plt.xlabel('$f$ [Hz]'); plt.ylabel('$|X_k|,|Y_k|$')
```

Bibliography

[1] R. E. Ziemer, W. H. Tranter, and D. R. Fannin, *Signals & systems – continuous and discrete*, 4th ed. (Prentice Hall, 1998).

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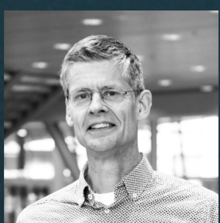

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